Neodređeni, određeni, nesvojstveni integrali i primene

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Sadržaj

Neka je R racionalna funkcija sa dve promenljive.
Ako važi da je $R(\sin x, -\cos x) = -R(\sin x, \cos x)$, uvodimo smenu $t = \sin x$. Razmatramo dva slučaja. Neka je $\cos x \geq 0$. Tada je $\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-t^2}$. Imamo da je $d\,t = \cos x\, d\,x$, odnosno da je $d\,x = \dfrac{d\,t}{\sqrt{1-t^2}}$. Prema tome,

$$\int R(\sin x, \cos x)\, d\,x = \int R\left(t, \sqrt{1-t^2}\right) \frac{d\,t}{\sqrt{1-t^2}}.$$

U slučaju da je $\cos x < 0$, imamo da je $\cos x = -\sqrt{1-\sin^2 x}$ $= -\sqrt{1-t^2}$ i da je $d\,x = -\dfrac{d\,t}{\sqrt{1-t^2}}$, odakle dobijamo da je

$$\begin{aligned} \int R(\sin x, \cos x)\, d\,x &= \int -R(\sin x, -\cos x)\, d\,x \\ &= \int -R\left(t, \sqrt{1-t^2}\right)\left(-\frac{d\,t}{\sqrt{1-t^2}}\right) \\ &= \int R\left(t, \sqrt{1-t^2}\right) \frac{d\,t}{\sqrt{1-t^2}}. \end{aligned}$$

Ako važi da je $R(-\sin x, \cos x) = -R(\sin x, \cos x)$, uvodimo smenu $t = \cos x$, $d\,t = -\sin x\; d\,x$. Ne gubeći na opštosti možemo uzeti da je $\sin x = \sqrt{1-t^2}$ i $d\,x = -\frac{d\,t}{\sqrt{1-t^2}}$. Dobijamo

$$\int R(\sin x, \cos x)\, d\,x = -\int R\left(\sqrt{1-t^2}, t\right)\frac{d\,t}{\sqrt{1-t^2}}.$$

Ako važi da je $R(-\sin x, -\cos x) = R(\sin x, \cos x)$, uvodimo smenu $t = \operatorname{tg} x$, $d\,t = \frac{d\,x}{\cos^2 x}$. Iz osnovnog trigonometrijskog identiteta $\sin^2 x + \cos^2 x = 1$, deljenjem sa $\sin^2 x$, odnosno $\cos^2 x$ dobijamo $1 + \frac{1}{\operatorname{tg}^2 x} = \frac{1}{\sin^2 x}$ i $\operatorname{tg}^2 x + 1 = \frac{1}{\cos^2 x}$, tj. imamo da je $\sin^2 x = \frac{\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x}$ i $\cos^2 x = \frac{1}{1+\operatorname{tg}^2 x}$. Ne gubeći na opštosti, zbog uslova $R(-\sin x, -\cos x) = R(\sin x, \cos x)$ možemo uzeti da je $\sin x = \frac{\operatorname{tg} x}{\sqrt{1+\operatorname{tg}^2 x}} = \frac{t}{\sqrt{1+t^2}}$ i $\cos x = \frac{1}{\sqrt{1+\operatorname{tg}^2 x}} = \frac{1}{\sqrt{1+t^2}}$, što posledično daje $d\,x = \frac{d\,t}{1+t^2}$. Dobijamo

$$\int R(\sin x, \cos x)\, d\,x = \int R\left(\frac{t}{\sqrt{1+t^2}}, \frac{1}{\sqrt{1+t^2}}\right)\frac{d\,t}{1+t^2}.$$

Ako funkcija $R(\sin x, \cos x)$ ne zadovoljava nijedan od prethodna tri uslova, uvodimo standardnu smenu $t = \operatorname{tg} \frac{x}{2}$. U ovom slučaju važi da je

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = \frac{2 \operatorname{tg} \frac{x}{2}}{1 + \operatorname{tg}^2 \frac{x}{2}} = \frac{2t}{1+t^2} \text{ i}$$

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \frac{1 - \operatorname{tg}^2 \frac{x}{2}}{1 + \operatorname{tg}^2 \frac{x}{2}} = \frac{1-t^2}{1+t^2}.$$

Kako je $dt = \frac{dx}{2\cos^2 \frac{x}{2}}$ imamo da je $dx = \frac{2\,dt}{1+t^2}$. Dobijamo

$$\int R(\sin x, \cos x)\,dx = 2 \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \frac{dt}{1+t^2}.$$

Zadatak

Odrediti sledeće integrale:

i) $\int \frac{\sin^2 x}{\cos^3 x}\, dx$,

ii) $\int \frac{dx}{(2+\cos x)\sin x}$,

iii) $\int \frac{\sin(2x)}{1+\cos^2 x}\, dx$,

iv) $\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$,

v) $\int \frac{dx}{\sin^3 x}$.

i) $$\int \frac{\sin^2 x}{\cos^3 x}\,dx = \int \frac{\sin^2 x}{\cos^4 x}\cos x\,dx = \left\{ \begin{array}{ccc} t & = & \sin x, \\ dt & = & \cos x\,dx \end{array} \right\} =$$
$$\int \frac{t^2\,dt}{\left(1-t^2\right)^2} = \left\{ \begin{array}{cc} u = t, & dv = \frac{t\,dt}{\left(t^2-1\right)^2} \\ du = dt, & v = -\frac{1}{2}\frac{1}{t^2-1} \end{array} \right\} =$$
$$-\frac{t}{2\left(t^2-1\right)} + \frac{1}{2}\int \frac{dt}{t^2-1} = -\frac{t}{2\left(t^2-1\right)} + \frac{1}{4}\ln\left|\frac{t-1}{t+1}\right| + C =$$
$$\frac{\sin x}{2\cos^2 x} + \frac{1}{4}\ln\left|\frac{\sin x - 1}{\sin x + 1}\right| + C$$

ii) $$\int \frac{dx}{(2+\cos x)\sin x} = \int \frac{\sin x\,dx}{(2+\cos x)\sin^2 x} =$$
$$\left\{ \begin{array}{ccc} t & = & \cos x, \\ dt & = & -\sin x\,dx \end{array} \right\} = -\int \frac{dt}{(2+t)\left(1-t^2\right)} =$$
$$\int \frac{dt}{(t+2)\left(t^2-1\right)}$$

$$\frac{1}{(t+2)\left(t^2-1\right)}=\frac{A}{t+2}+\frac{B}{t+1}+\frac{C}{t-1}=\frac{A\left(t^2-1\right)+B(t+2)(t-1)+C(t+2)(t+1)}{(t+2)\left(t^2-1\right)}$$

$$A\left(t^2-1\right)+B(t+2)(t-1)+C(t+2)(t+1)=1$$

$$t=1 \quad \Rightarrow \quad C=\tfrac{1}{6}$$

$$t=-1 \quad \Rightarrow \quad B=-\tfrac{1}{2}$$

$$t=-2 \quad \Rightarrow \quad A=\tfrac{1}{3}$$

$$\int\frac{dt}{(t+2)\left(t^2-1\right)}=\frac{1}{3}\int\frac{dt}{t+2}-\frac{1}{2}\int\frac{dt}{t+1}+\frac{1}{6}\int\frac{dt}{t-1}=$$

$$\frac{1}{6}\ln\left|\frac{(t+2)^2(t-1)}{(t+1)^3}\right|+C=\frac{1}{6}\ln\left|\frac{(\cos x+2)^2(\cos x-1)}{(\ cosx+1)^3}\right|+C$$

iii) $$\int\frac{\sin(2x)}{1+\cos^2 x}dx=\int\frac{2\sin x\cos x}{1+\cos^2 x}dx=$$

$$\left\{\begin{array}{rcl} t & = & \cos x, \\ dt & = & -\sin x\,dx \end{array}\right\}=-\int\frac{2t\,dt}{1+t^2}=-\ln\left(1+t^2\right)+C=$$

$$-\ln\left(1+\cos^2 x\right)+C$$

iv) $\displaystyle\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\,dx, = \int \frac{\frac{\sin x}{\cos x}}{\frac{\sin^4 x}{\cos^4 x} + 1}\,\frac{dx}{\cos^2 x} ==$

$$\left\{ \begin{array}{rcl} t & = & \operatorname{tg} x, \\ dt & = & \frac{dx}{\cos^2 x} \end{array} \right\} = \int \frac{t\,dt}{t^4+1} = \frac{1}{2}\int \frac{dt^2}{\left(t^2\right)^2+1} =$$

$$\frac{1}{2}\operatorname{arctg} t^2 + C = \frac{1}{2}\operatorname{arctg}\left(\operatorname{tg}^2 x\right) + C$$

v) $\displaystyle\int \frac{dx}{\sin^3 x} = \left\{ \begin{array}{rr} t = \operatorname{tg}\frac{x}{2}, & dx = \frac{2\,dt}{1+t^2} \\ dt = \frac{dx}{2\cos^2\frac{x}{2}}, & \sin x = \frac{2t}{1+t^2} \end{array} \right\} =$

$$\int \frac{\left(1+t^2\right)^2 dt}{4t^3} = \frac{1}{4}\int t\,dt + \frac{1}{2}\int \frac{dt}{t} + \frac{1}{4}\int \frac{dt}{t^3} =$$

$$\frac{t^2 - t^{-2}}{8} + \frac{\ln|t|}{2} + C = \frac{\operatorname{tg}^2\frac{x}{2} - \operatorname{tg}^{-2}\frac{x}{2}}{8} + \frac{\ln\left|\operatorname{tg}\frac{x}{2}\right|}{2} + C$$

Domaći zadatak

Odrediti sledeće integrale:

i) $\int \sin^2 x \cos^3 x,$ $\left[\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C\right]$

ii) $\int \frac{dx}{\sin^2 x \cos x},$ $\left[\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| - \frac{1}{\sin x} + C\right]$

iii) $\int \frac{\sin x}{\cos^2 x}\sqrt{1+2\cos^2 x}\,dx,$ $\left[\sqrt{\cos^{-2} x + 2} + \sqrt{2}\ln\left|\sqrt{1+2\cos^2 x} - \sqrt{2}\cos x\right| + C\right]$

iv) $\int \sin^5 x \cos^6 x\,dx,$ $\left[-\frac{\cos^{11} x}{11} + \frac{2\cos^9 x}{9} - \frac{\cos^7 x}{7} + C\right]$

v) $\int \frac{\sin^2 x + \cos x}{\sin^2 x - \cos x}\sin x\,dx,$ $\left[\ln\left|\cos^2 x + \cos x - 1\right| - \frac{1}{\sqrt{5}}\ln\left|\frac{2\cos x - \sqrt{5}+1}{2\cos x + \sqrt{5}+1}\right| - \cos x + C\right]$

vi) $\int \frac{2\sin x - \cos x}{3\sin^2 x + 4\cos^2 x}\,dx,$ $\left[\frac{1}{4}\ln\left|\frac{2-\sin x}{2+\sin x}\right| - \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{\cos x}{\sqrt{3}} + C\right]$

vii) $\int \frac{dx}{\cos(2x)},$ $\left[\frac{1}{2}\ln\left|\frac{\sin(2x)+1}{\cos(2x)}\right| + C\right]$

Domaći zadatak

viii) $\int \frac{\sin x \cos x}{1+\sin^4 x} dx,$ $\left[\frac{arctg\left(\sin^2 x\right)}{2} + C\right]$

ix) $\int \frac{\sin x \, dx}{\cos^3 x + \sin^3 x},$ $\left[-\frac{1}{3}\ln|tg\, x+1| + \frac{1}{6}\ln\left(tg^2 x - tg\, x + 1\right) + \frac{1}{\sqrt{3}} arctg\, \frac{2\, tg\, x - 1}{\sqrt{3}} + C\right]$

x) $\int \frac{\sin^2 x}{1+\sin^2 x} dx,$ $\left[x - \frac{1}{\sqrt{2}} arctg\left(\sqrt{2} tg\, x\right) + C\right]$

xi) $\int \frac{\sin^2 x}{\cos^6 x} dx,$ $\left[\frac{1}{5} tg^5 x + \frac{1}{3} tg^3\, x + C\right]$

xii) $\int \sin^4 x \, dx,$ $\left[\frac{1}{32}\sin(4x) - \frac{1}{4}\sin(2x) + \frac{3x}{8} + C\right]$

xiii) $\int \frac{dx}{2\sin x - \cos x + 5},$ $\left[\frac{1}{\sqrt{5}} arctg\, \frac{3\, tg\, \frac{x}{2} + 1}{\sqrt{5}} + C\right]$

xiv) $\int \frac{1 - \sin x + \cos x}{1 + \sin x - \cos x} dx.$ $\left[-x - 2\ln\left|\frac{tg\, \frac{x}{2} + 1}{tg\, \frac{x}{2}}\right| + C\right]$

- $$\int \sin(\alpha x)\cos(\beta x)\,dx = \frac{1}{2}\int (\sin((\alpha+\beta)x)+\sin((\alpha-\beta)x))\,dx = -\frac{1}{2}\left(\frac{\cos((\alpha+\beta)x)}{\alpha+\beta}+\frac{\cos((\alpha-\beta)x)}{\alpha-\beta}\right)+C,\ \alpha\neq\pm\beta,$$
- $$\int \cos(\alpha x)\cos(\beta x)\,dx = \frac{1}{2}\int (\cos((\alpha+\beta)x)+\cos((\alpha-\beta)x))\,dx = \frac{1}{2}\left(\frac{\sin((\alpha+\beta)x)}{\alpha+\beta}+\frac{\sin((\alpha-\beta)x)}{\alpha-\beta}\right)+C,\ \alpha\neq\pm\beta,$$
- $$\int \sin(\alpha x)\sin(\beta x)\,dx = \frac{1}{2}\int (-\cos((\alpha+\beta)x)+\cos((\alpha-\beta)x))\,dx = \frac{1}{2}\left(\frac{-\sin((\alpha+\beta)x)}{\alpha+\beta}+\frac{\sin((\alpha-\beta)x)}{\alpha-\beta}\right)+C,\ \alpha\neq\pm\beta.$$

Zadatak

Odrediti sledeće integrale:

i) $\int \sin(2x)\cos(3x)\,dx$,

ii) $\int \sin x \sin(2x)\sin(3x)\,dx$,

iii) $\int \cos x \cos(2x)\cos(3x)\,dx$.

Rešenje:

$$\int \sin x \sin(2x)\sin(3x)\,dx, =$$
$$\frac{1}{2}\int \sin(2x)\left(-\cos(4x)+\cos(2x)\right)dx =$$
$$\frac{1}{4}\int \left(-\sin(6x)+\sin(2x)+\sin(4x)\right)dx =$$
$$\frac{1}{4}\left(\frac{\cos(6x)}{6}-\frac{\cos(2x)}{2}-\frac{\cos(4x)}{4}\right)+C.$$

Funkcija f definisana na segmentu $[a, b]$ je integrabilna na tom segmentu ako i samo ako je ograničena na $[a, b]$ i ako ima najviše prebrojivo tačaka prekida na $[a, b]$.

Njutn-Lajbnicova formula:

Ako je funkcija f neprekidna na segmentu $[a, b]$, onda važi

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a),$$

gde je F primitivna funkcija funkcije f na $[a, b]$.

Zadatak

Izračunati sledeće određene integrale:

$$i)\ \int_0^{\pi} sinx\, dx,$$

$$ii)\ \int_{-1}^{1} \frac{1}{x}\, dx,$$

$$iii)\ \int_0^{2} f(x)\, dx, \quad f(x) = \begin{cases} x^2,\ 0 \leq x \leq 1, \\ 2 - x,\ 1 \leq x \leq 2, \end{cases}$$

$$iv)\ \int_0^{2} \sqrt{x^2 - 2x + 1}\, dx.$$

Rešenje:

i) $$\int_0^{\pi} sinx\,dx = -cosx\Big|_0^{\pi} = -cos(\pi) - (-cos(0)) = 2.$$

ii) Podintegralna funkcija $f(x) = \frac{1}{x}$ nije ograničena na segmentu $[-1, 1]$, pa prema tome nije integrabilna u Rimanovom smislu.

iii) $$\int_0^2 f(x)\,dx, \quad f(x) = \begin{cases} x^2, \; 0 \leq x \leq 1, \\ 2 - x, \; 1 \leq x \leq 2, \end{cases}$$

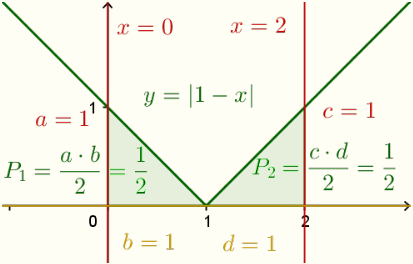
$$\int_0^2 f(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 (2-x)\,dx =$$

$$\frac{x^3}{3}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^2 = \frac{1}{3} + 4 - 2 - \left(2 - \frac{1}{2}\right) = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

$$iv)\ \int_0^2 \sqrt{x^2-2x+1}\,dx = \int_0^2 |1-x|\,dx =$$

$$\int_0^1 (1-x)\,dx + \int_1^2 (x-1)\,dx = \left(x-\frac{x^2}{2}\right)\Big|_0^1 + \left(\frac{x^2}{2}-x\right)\Big|_1^2 =$$

$$1-\frac{1}{2}+2-2-\left(\frac{1}{2}-1\right)=1$$



Metoda parcijalne integracije u određenom integralu:

Neka funkcije u i v imaju neprekidne prve izvode na segmentu $[a, b]$. Tada je

$$\begin{aligned}\int_a^b u(x)\,v'(x)\,dx &= u(x)\,v(x)\Big|_a^b - \int_a^b v(x)\,u'(x)\,dx \\ &= u(b)\,v(b) - u(a)\,v(a) - \int_a^b v(x)\,u'(x)\,dx.\end{aligned}$$

Zadatak

Izračunati određeni integral $\int\limits_0^{\sqrt{3}} \arcsin \frac{2x}{1+x^2}\, dx.$

Rešenje: Neka je $u = \arcsin \frac{2x}{1+x^2}$ i $v = x$. Tada je

$u' = \frac{1}{\sqrt{1-\frac{4x^2}{(1+x^2)^2}}} \frac{2(1+x^2)-4x^2}{(1+x^2)^2} = \frac{2(1-x^2)}{|1-x^2|(1+x^2)}$ i $v' = 1$.

Funkcija u' nije neprekidna na segmentu $[0, \sqrt{3}]$, ima prekid u tački $x = 1$, prema tome, ne možemo primeniti metodu parcijalne integracije. Međutim, ova funkcija je neprekidna na intervalima $[0, 1)$ $(1, \sqrt{3}]$.

$$\int\limits_0^{\sqrt{3}} \arcsin \frac{2x}{1+x^2}\, dx = \int\limits_0^{1} \arcsin \frac{2x}{1+x^2}\, dx + \int\limits_1^{\sqrt{3}} \arcsin \frac{2x}{1+x^2}\, dx =$$

$$= \left\{ \begin{array}{c} u = \arcsin\frac{2x}{1+x^2}, \quad du = \frac{2(1-x^2)}{|1-x^2|(1+x^2)}, \\ dv = dx, \quad v = x \end{array} \right\} =$$

$$= x\arcsin\frac{2x}{1+x^2}\Big|_0^1 - \int_0^1 \frac{2xdx}{(1+x^2)} + x\arcsin\frac{2x}{1+x^2}\Big|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{2xdx}{(1+x^2)} =$$

$$= x\arcsin\frac{2x}{1+x^2}\Big|_0^{\sqrt{3}} - ln(1+x^2)\Big|_0^1 + ln(1+x^2)\Big|_1^{\sqrt{3}} =$$

$$= \sqrt{3}\,\frac{\pi}{3} - ln(2) + ln(4) - ln(2) = \sqrt{3}\,\frac{\pi}{3}$$

Koristili smo da je

$$|1-x^2| = \left\{ \begin{array}{l} x^2-1, \; x<-1 \vee x>1, \\ 1-x^2, \; -1<x<1. \end{array} \right.$$

Smena $\varphi(x) = t$

$$\int_a^b f(\varphi(x))\,dx = \int_\alpha^\beta f(t)\psi'(t)\,dt$$

- f neprekidna funkcija na segmentu $[\alpha, \beta]$
- $\alpha = \varphi(a), \beta = \varphi(b)$;
- φ je strogo monotona funkcija na segmentu $[a, b]$;
- $\psi = \varphi^{-1}, \psi'$ je neprekidna funkcija na segmentu $[\alpha, \beta]$.

Smena $x = \varphi(t)$

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$$

- f neprekidna funkcija na segmentu $[a, b]$;
- $a = \varphi(\alpha), b = \varphi(\beta)$;
- φ' neprekidna funkcija na segmentu $[\alpha, \beta]$;
- $f \circ \varphi$ je definisana za svako $x \in [\alpha, \beta]$.

Zadatak

Izračunati određene integrale:

i) $\int\limits_0^4 x\sqrt{x^2+9}\,dx$,

ii) $\int\limits_{-4}^4 x\sqrt{x^2+9}\,dx$.

Rešenje: Uvedimo smenu $t = x^2+9$, funkcija $\varphi(x) = x^2+9$ je monotono rastuća na segmentu $[0,4]$. Imamo da je $dt = 2x\,dx$ i da je za $x = 0$, $t = 9$, a za $x = 4$, $t = 25$. Prema tome,

$$\int\limits_0^4 x\sqrt{x^2+9}\,dx = \frac{1}{2}\int\limits_9^{25}\sqrt{t}\,dt = \frac{1}{3}\sqrt{t^3}\Big|_9^{25} = \frac{125-27}{3} = \frac{98}{3}.$$

Kako je $f(x) = x\sqrt{x^2+9}$ neparna funkcija i kao je segment $[-4,4]$ simetričan u odnosu na koordinatni početak važi da je

$$\int\limits_{-4}^4 x\sqrt{x^2+9}\,dx = 0.$$

Zadatak

Izračunati određeni integral $\int\limits_0^3 \arcsin\sqrt{\frac{x}{1+x}}\, dx$.

Rešenje: Primenimo metodu parcijalne integracije. Neka je $u = \arcsin\sqrt{\frac{x}{1+x}}$. Tada je $u' = \frac{1}{\sqrt{1-\frac{x}{1+x}}}\frac{1}{2}\sqrt{\frac{1+x}{x}}\frac{1+x-x}{(1+x)^2} = \frac{1}{2}\frac{1}{\sqrt{x}(1+x)}$.
Dalje, imamo $v = x$ i $v' = 1$. Prema tome,

$$\int\limits_0^3 \arcsin\sqrt{\frac{x}{1+x}}\, dx = \left\{ \begin{array}{cc} u = \arcsin\sqrt{\frac{x}{1+x}}, & du = \frac{1}{2}\frac{dx}{\sqrt{x}(1+x)}, \\ dv = dx, & v = x \end{array} \right\} =$$

$$x \arcsin\sqrt{\frac{x}{1+x}}\Bigg|_0^3 - \frac{1}{2}\int\limits_0^3 \frac{x\, dx}{\sqrt{x}(1+x)} = \pi - \frac{1}{2}\int\limits_0^3 \frac{x\, dx}{\sqrt{x}(1+x)}.$$

Uvedimo sada smenu $t = \sqrt{x}$ u integral $\dfrac{1}{2}\int\limits_0^3 \dfrac{x\,dx}{\sqrt{x}(1+x)}$.

Funkcija $\varphi(x) = \sqrt{x}$ je strogo monotono rastuća funkcija na segmentu $[0,3]$.

Važi $\dfrac{1}{2}\int\limits_0^3 \dfrac{x\,dx}{\sqrt{x}(1+x)} = \left\{ \begin{array}{rl} t = \sqrt{x}, & x = 0 \to t = 0, \\ d\,t = \frac{1}{2}\frac{d\,x}{\sqrt{x}}, & x = 3 \to t = \sqrt{3} \end{array} \right\} =$

$$\int\limits_0^{\sqrt{3}} \frac{t^2 dt}{1+t^2} = (t - \operatorname{arctg} t)\Big|_0^{\sqrt{3}} = \sqrt{3} - \frac{\pi}{3}.$$

Zaključujemo $\displaystyle\int\limits_0^3 \arcsin\sqrt{\frac{x}{1+x}}\,d\,x = \frac{4}{3}\pi - \sqrt{3}$.

Domaći zadatak

Izračunati određene integrale:

$i)$ $\int_0^{\sqrt{3}} x \arcsin \frac{2x}{1+x^2}\, dx,$ $\left[\frac{2}{3}\pi + \sqrt{3} - 2\right]$

$ii)$ $\int_0^1 2x \operatorname{arctg} \sqrt{x}\, dx,$ $\left[\frac{2}{3}\right]$

$iii)$ $\int_1^2 x \operatorname{arctg} \frac{2x}{x^2-1}\, dx,$ $\left[\operatorname{arctg} \frac{4}{3} - \operatorname{arctg} 2 + 1\right]$

$iv)$ $\int_0^1 \frac{\operatorname{arctg} \sqrt[3]{x}}{1+\sqrt[3]{x^2}}\, dx,$ $\left[\frac{3}{4}\pi - \frac{3}{32}\pi^2 - \frac{3}{2}\ln 2\right]$

$v)$ $\int_0^2 \ln \frac{x+4}{4-x}\, dx,$ $\left[-8\ln 2 + 6\ln 3\right]$

Domaći zadatak

vi) $\displaystyle\int_0^1 \frac{\ln(1+x^2)}{(1+x)^2}\,dx,$ $\quad\left[\frac{1}{4}\pi-\ln 2\right]$

vii) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\ln(\cos x)}{(\sin x+\cos x)^2}\,dx,$ $\quad\left[\frac{\ln 2}{2}-\frac{\pi}{8}\right]$

viii) $\displaystyle\int_0^{2\pi} e^x|\sin x|\,dx,$ $\quad\left[\frac{1}{2}(1+e^{\pi})^2\right]$

ix) $\displaystyle\int_0^{\ln 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}\,dx,$ $\quad[4-\pi]$

x) $\displaystyle\int_0^{\ln 2} \frac{x\,e^x}{(1+e^x)^2}\,dx.$ $\quad\left[\frac{5}{3}\ln 2-\ln 3\right]$

Zadatak

i) Neka je funkcija f neprekidna na segmentu $[-a, a]$ i neparna. Tada je $\int_{-a}^{a} f(x)\,dx = 0$.

ii) Neka je funkcija f neprekidna na segmentu $[-a, a]$ i parna. Tada je $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

Rešenje: $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

$$\int_{-a}^{0} f(x)\,dx = \left\{ \begin{array}{rl} x = -t, & x = -a \to t = a, \\ dx = -dt, & x = 0 \to t = 0 \end{array} \right\} = -\int_{a}^{0} f(-t)\,dt =$$

$$\int_{a}^{0} f(t)\,dt = -\int_{0}^{a} f(t)\,dt = -\int_{0}^{a} f(x)\,dx$$

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$$

Domaći zadatak

Izračunati određene integrale:

$i)$ $$\int_{-1}^{1} \frac{x^{17}}{\sqrt{1+x^2}}\, dx,$$

$ii)$ $$\int_{-\pi}^{\pi} e^{x^2} \sin(2x) \cos^{\frac{4}{3}} x\, dx,$$

$iii)$ $$\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos x \ln \frac{1+x}{1-x}\, dx,$$

$iv)$ $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \operatorname{tg} x\, dx.$$

Zadatak

Izvesti rekurentnu formulu za određeni integral $\int\limits_0^{\frac{\pi}{2}} sin^n x\, d\, x.$

Rešenje: Primenimo metodu parcijalne integracije

$$I_n = \int\limits_0^{\frac{\pi}{2}} sin^n x\, d\, x = \left\{ \begin{array}{ll} u = \sin^{n-1} x, & d\, u = (n-1)\sin^{n-2} x \cos x d\, x, \\ d\, v = \sin x d\, x, & v = -\cos x \end{array} \right\}$$

$$= -\cos x \sin^{n-1} x \Big|_0^{\frac{\pi}{2}} + (n-1) \int\limits_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x\, d\, x$$

$$= (n-1) \left(\int\limits_0^{\frac{\pi}{2}} \sin^{n-2} x\, d\, x - \int\limits_0^{\frac{\pi}{2}} \sin^n d\, x \right)$$

$$= (n-1) I_{n-2} - (n-1) I_n.$$

Dobijamo rekurentnu formulu $I_n = \frac{n-1}{n} I_{n-2}$. Odakle sledi da je za $n = 2k$

$$I_{2k} = \frac{2k-1}{2k} I_{2k-2} = \frac{2k-1}{2k}\frac{2k-3}{2k-2} I_{2k-4} = \ldots = \frac{2k-1}{2k}\frac{2k-3}{2k-2}\cdots\frac{1}{2} I_0$$

$$I_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}, \quad I_{2k} = \frac{(2k-1)!!}{(2k)!!}\frac{\pi}{2},$$

a za $n = 2k+1$

$$I_{2k+1} = \frac{2k}{2k+1} I_{2k-1} = \frac{2k}{2k+1}\frac{2k-2}{2k-1} I_{2k-3} = \ldots = \frac{2k}{2k+1}\frac{2k-2}{2k-1}\cdots\frac{2}{3} I_1$$

$$I_1 = \int_0^{\frac{\pi}{2}} sinx\, dx, = -cosx \,|_0^{\frac{\pi}{2}} = 1, \quad I_{2k+1} = \frac{(2k)!!}{(2k+1)!!}.$$

Domaći zadatak

Izvesti rekurentne formule za izračunavanje određenih integrala:

i) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^n x\,dx,$ $\left[I_n = \frac{n-1}{n} I_{n-2}\right]$

ii) $\displaystyle\int_0^{\frac{\pi}{4}} \operatorname{tg}^{2n} x\,dx,$ $\left[I_n = \frac{1}{2n-1} - I_{n-1}\right]$

iii) $\displaystyle\int_0^{1} \frac{x^n}{\sqrt{1-x^2}}\,dx,$ $\left[I_n = \frac{n-1}{n} I_{n-2}\right]$

iv) $\displaystyle\int_{-1}^{1} (1-x^2)^n\,dx.$ $\left[I_n = \frac{2n}{2n+1} I_{n-1}\right]$

Neka je funkcija f definisana na intervalu $[a, +\infty)$ i neka je integrabilna na svakom segmentu $[a, b]$. Ako postoji $\lim\limits_{b \to +\infty} F(b)$ i konačan je, gde je F primitivna funkcija funkcije f na intervalu $[a, +\infty)$, onda kažemo da je $\int\limits_a^{+\infty} f(x)\,dx$ konvergentan, u suprotnom je divergentan.

$$\int\limits_a^{+\infty} f(x)\,dx = \lim_{b \to +\infty} \int\limits_a^{b} f(x)\,dx = \lim_{b \to +\infty} F(b) - F(a)$$

Zadatak

Ispitati konvergenciju integrala $\int\limits_0^{+\infty} \frac{dx}{x^2+4x+5}$.

$$\int\limits_0^{+\infty} \frac{dx}{x^2+4x+5} = \int\limits_0^{+\infty} \frac{dx}{(x+2)^2+1} = \operatorname{arctg}(x+2)\Big|_0^{+\infty}$$

$$= \lim_{b \to +\infty} \operatorname{arctg}(b+2) - \operatorname{arctg}(2) = \frac{\pi}{2} - \operatorname{arctg}(2)$$

Zadatak

Ispitati konvergenciju integrala $\int\limits_2^{+\infty} \frac{dx}{x^2+x-2}$.

$$\begin{aligned}
\int\limits_2^{+\infty} \frac{dx}{x^2+x-2} &= \int\limits_2^{+\infty} \frac{dx}{(x+2)(x-1)} \\
&= \frac{1}{3}\int\limits_2^{+\infty} \frac{x+2-(x-1)}{(x+2)(x-1)}\,dx \\
&= \frac{1}{3}\int\limits_2^{+\infty} \left(\frac{1}{x-1} - \frac{1}{x+2}\right) dx \\
&= \frac{1}{3}\ln\frac{x-1}{x+2}\Big|_2^{+\infty} \\
&= \lim_{b\to+\infty} \frac{1}{3}\ln\frac{b-1}{b+2} - \frac{1}{3}\ln\frac{2-1}{2+2} = \frac{\ln 4}{3}
\end{aligned}$$

Zadatak

Ispitati konvergenciju integrala $\int\limits_1^{+\infty} \frac{\operatorname{arctg} x}{x^2}\, dx$.

$$\begin{aligned}
\int \frac{\operatorname{arctg} x}{x^2}\, dx &= \left\{ \begin{array}{cc} u = \operatorname{arctg} x, & dv = \frac{1}{x^2}, \\ du = \frac{dx}{1+x^2}, & v = -\frac{1}{x} \end{array} \right\} \\
&= -\frac{\operatorname{arctg} x}{x} + \int \frac{dx}{x(1+x^2)} \\
&= -\frac{\operatorname{arctg} x}{x} + \int \frac{x^2 + 1 - x^2}{x(1+x^2)}\, dx \\
&= -\frac{\operatorname{arctg} x}{x} + \int \frac{dx}{x} - \int \frac{x}{1+x^2}\, dx \\
&= -\frac{\operatorname{arctg} x}{x} + \ln x - \frac{1}{2}\ln(1+x^2) \\
&= -\frac{\operatorname{arctg} x}{x} + \ln \frac{x}{\sqrt{1+x^2}}
\end{aligned}$$

$$\begin{aligned}\int_1^{+\infty} \frac{\operatorname{arctg} x}{x^2}\,dx &= \lim_{b\to+\infty}\int_1^{b} \frac{\operatorname{arctg} x}{x^2}\,dx \\ &= \lim_{b\to+\infty}\left(-\frac{\operatorname{arctg} b}{b} + \ln\frac{b}{\sqrt{1+b^2}}\right) + \frac{\operatorname{arctg} 1}{1} - \ln\frac{1}{\sqrt{1+1^2}} \\ &= \frac{\pi}{4} + \ln\sqrt{2}\end{aligned}$$

Zadatak

Izračunati integral $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{3\sin^2 x + 5\cos^2 x}$.

$$\int_0^{\frac{\pi}{2}} \frac{dx}{3\sin^2 x + 5\cos^2 x} = \int_0^{\frac{\pi}{2}} \frac{1}{3\operatorname{tg}^2 x + 5}\frac{dx}{\cos^2 x} =$$

$$\left\{\begin{matrix} t = \operatorname{tg} x, & x = 0 \to t = 0, \\ dt = \frac{dx}{\cos^2 x}, & x = \frac{\pi}{2} \to t = +\infty \end{matrix}\right\} = \int_0^{+\infty} \frac{dt}{3t^2+5} = \frac{1}{\sqrt{15}}\operatorname{arctg}\left(\sqrt{\frac{3}{5}}\,t\right)\Big|_0^{+\infty} =$$

$$\lim_{b\to+\infty}\frac{1}{\sqrt{15}}\operatorname{arctg}\left(\sqrt{\frac{3}{5}}\,b\right) - \operatorname{arctg}(0) = \frac{\pi}{2\sqrt{15}}$$

Zadatak

Izračunati integral $\int\limits_0^{2\pi} \frac{dx}{5+2\cos x}$.

Integral $\int\limits_0^{2\pi} \frac{dx}{5+2\cos x}$ ne možemo rešavati smenom $t = \operatorname{tg} \frac{x}{2}$, jer funkcija $\varphi(x) = \operatorname{tg} \frac{x}{2}$ ima prekid u tački $x = \pi$. Važi da je $\int\limits_0^{2\pi} \frac{dx}{5+2\cos x} = \int\limits_0^{\pi} \frac{dx}{5+2cosx} + \int\limits_{\pi}^{2\pi} \frac{dx}{5+2\cos x}$. Na segmentima $[0,\pi]$ i $[\pi,2\pi]$ funkcija $\varphi(x) = \operatorname{tg} \frac{x}{2}$ je rastuća funkcija, sledi

$$\int\limits_0^{2\pi} \frac{dx}{5+2cosx} = \int\limits_0^{\pi} \frac{dx}{5+2cosx} + \int\limits_{\pi}^{2\pi} \frac{dx}{5+2cosx} =$$

$$\left\{ \begin{array}{lll} t = \operatorname{tg} \frac{x}{2}, & x = 0 \to t = 0, & x = \pi_+ \to t = -\infty, \\ dx = \frac{2dt}{1+t^2}, & x = \pi_- \to t = +\infty, & x = 2\pi \to t = 0 \end{array} \right\} =$$

$$\int_0^{+\infty} \frac{2\,dt}{\left(5+2\frac{1-t^2}{1+t^2}\right)(1+t^2)} + \int_{-\infty}^{0} \frac{2\,dt}{\left(5+2\frac{1-t^2}{1+t^2}\right)(1+t^2)} =$$

$$\int_0^{+\infty} \frac{2\,dt}{7+3t^2} + \int_{-\infty}^{0} \frac{2\,dt}{7+3t^2} = \int_{-\infty}^{+\infty} \frac{2\,dt}{7+3t^2} = \frac{2}{7}\int_{-\infty}^{+\infty} \frac{dt}{1+\left(\sqrt{\frac{3}{7}}\,t\right)^2} =$$

$$\left\{\begin{array}{rl} u = \sqrt{\frac{3}{7}}\,t, & t=-\infty \to u=-\infty, \\ du = \sqrt{\frac{3}{7}}\,dt, & t=+\infty \to u=+\infty \end{array}\right\} =$$

$$\frac{2}{\sqrt{21}}\int_{-\infty}^{+\infty} \frac{du}{1+u^2} = \frac{2}{\sqrt{21}} \operatorname{arctg} u \Big|_{-\infty}^{+\infty} =$$

$$\frac{2}{\sqrt{21}}\left(\lim_{b\to+\infty} \operatorname{arctg}(b) - \lim_{a\to-\infty} \operatorname{arctg}(a)\right) = \frac{2}{\sqrt{21}}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{2\pi}{\sqrt{21}}.$$

Domaći zadatak

Ispitati konvergenciju integrala

i) $\displaystyle\int_0^{+\infty} \frac{dx}{x^2}$, [1]

ii) $\displaystyle\int_0^{+\infty} \frac{dx}{x}$, [divergira]

iii) $\displaystyle\int_0^{+\infty} \cos x \, dx$, [divergira]

iv) $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$, [$\pi$]

v) $\displaystyle\int_0^{+\infty} \frac{\operatorname{arctg} x}{(1+x)^2} \, dx$. [$\frac{\pi}{4}$]

Neka je funkcija f neograničena u okolini tačke b i neka je integrabilna na svakom segmentu $[a, b-\varepsilon]$. Tada važi

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0_+} \int_a^{b-\varepsilon} f(x)\,dx.$$

Zadatak

Ispitati konvergenciju integrala $\displaystyle\int_{-1}^{1} \frac{dx}{x}$.

$$\int_{-1}^{1} \frac{dx}{x} = \int_{-1}^{0} \frac{dx}{x} + \int_{0}^{1} \frac{dx}{x} = \lim_{\varepsilon \to 0_+} \int_{-1}^{-\varepsilon} \frac{dx}{x} + \lim_{\xi \to 0_+} \int_{\xi}^{1} \frac{dx}{x} =$$

$$\lim_{\varepsilon \to 0_+} \ln|\varepsilon| - \ln|-1| + \ln|1| - \lim_{\xi \to 0_+} \ln|\xi| = \lim_{\varepsilon \to 0_+} \ln|\varepsilon| - \lim_{\xi \to 0_+} \ln|\xi| = \infty - \infty$$

Integral divergira. Podintegralna funkcija je neparna i neograničena u nuli, ali integrabilna na svakom segmentu koji ne sadrži nulu, pa je $v.p. \displaystyle\int_{-1}^{1} \frac{dx}{x} = 0.$

Zadatak

Ispitati konvergenciju integrala $\int\limits_0^1 \ln x \, dx$.

$$\int \ln x \, dx = \left\{ \begin{array}{cc} u = \ln x & du = \frac{dx}{x} \\ dv = dx & v = x \end{array} \right\} = x \ln x - \int dx =$$
$$x \ln x - x + C = x(\ln x - 1) + C$$

$$\int\limits_0^1 \ln x \, dx = \lim_{\varepsilon \to 0_+} \int\limits_\varepsilon^1 \ln x \, dx =$$
$$-1 - \lim_{\varepsilon \to 0_+} (\varepsilon(\ln \varepsilon - 1)) = -1 - \lim_{\varepsilon \to 0_+} \frac{\ln \varepsilon - 1}{\frac{1}{\varepsilon}} =$$
$$-1 - \lim_{\varepsilon \to 0_+} \frac{\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}} = -1 + \lim_{\varepsilon \to 0_+} \varepsilon = -1$$

Domaći zadatak

Ispitati konvergenciju integrala

i) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$, [2]

ii) $\displaystyle\int_1^2 \frac{dx}{x\ln x}$, [divergira]

iii) $\displaystyle\int_{-\frac{\pi}{2}}^0 \sin\frac{1}{x}\,\frac{dx}{x^2}$. [divergira]

Zadatak

Izračunati integral $\displaystyle\int_0^1 \frac{x\,e^{\arcsin x}}{\sqrt{1-x^2}}\,dx$.

Podintegralna funkcija nije ograničena u okolini tačke $x = 1$. U pitanju je nesvojstveni integral II vrste. Uvodimo smenu $t = \arcsin x$, $dt = \frac{dx}{\sqrt{1-x^2}}$. Funkcija $\varphi(x) = \arcsin x$ je rastuća funkcija i važi da je za $t = \varphi(0) = 0$ i $t = \varphi(1) = \frac{\pi}{2}$. Imamo da je

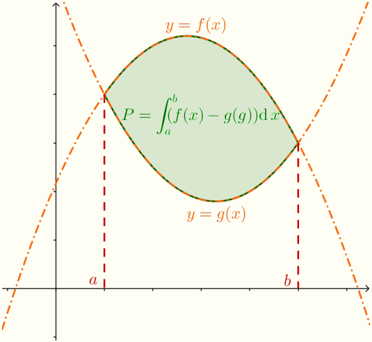
$$\int_0^1 \frac{x\, e^{\arcsin x}}{\sqrt{1-x^2}}\, dx = \int_0^{\frac{\pi}{2}} \sin t\, e^t dt.$$

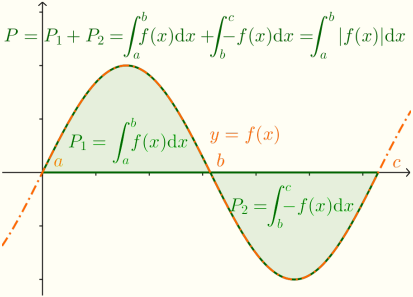
Integral sa desne strane je određen integral koji rešavamo metodom parcijalne integracije.

$$\begin{aligned} I &= \int \sin t\, e^t dt = \left\{ \begin{array}{cc} u = \sin t, & du = \cos t\, dt, \\ dv = e^t dt, & v = e^t \end{array} \right\} \\ &= \sin t\, e^t - \int \cos t\, e^t dt = \left\{ \begin{array}{cc} u = \cos t, & du = -\sin t\, dt, \\ dv = e^t dt, & v = e^t \end{array} \right\} \\ &= \sin t\, e^t - \cos t\, e^t - \int \sin t\, e^t dt = (\sin t - \cos t)\, e^t - I \end{aligned}$$

$$I = \frac{1}{2}(\sin t - \cos t)\, e^t + C$$

$$\int_0^{\frac{\pi}{2}} \sin t\, e^t dt = \frac{1}{2}(\sin t - \cos t)\, e^t \Big|_0^{\frac{\pi}{2}} = \frac{1}{2} e^{\frac{\pi}{2}} + \frac{1}{2}$$


y = f(x)
P = ∫_a^b (f(x) − g(g)) d x
y = g(x)
a
b


P = P_1 + P_2 = ∫_a^b f(x)dx + ∫_b^c −f(x)dx = ∫_a^b |f(x)|dx
P_1 = ∫_a^b f(x)dx
y = f(x)
a
b
c
P_2 = ∫_b^c −f(x)dx

Zadatak

Izračunati veličinu površine ograničene krivom $y = \sin x$ i odsečkom x-ose $[0, 2\pi]$.

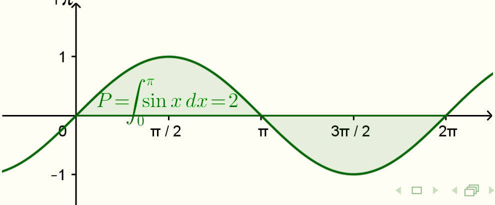
Važi da je

$$\int_0^{2\pi} \sin x \, dx = -\cos x \Big|_0^{2\pi} = -\cos(2\pi) + \cos(0) = -1 + 1 = 0,$$

dok je veličina tražene površine jednaka je

$$\int_0^{2\pi} |\sin x| \, dx = \int_0^{\pi} \sin x \, dx - \int_{\pi}^{2\pi} \sin x \, dx =$$

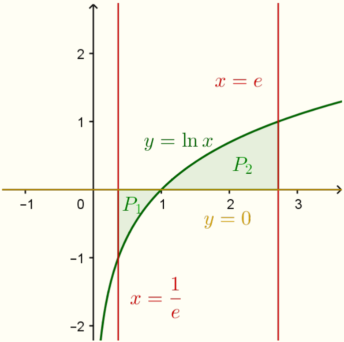
$$-\cos x \Big|_0^{\pi} + \cos x \Big|_{\pi}^{2\pi} = -\cos(\pi) + \cos(0) + \cos(2\pi) - \cos(\pi) = 4.$$



Zadatak

Izračunati veličinu površine ograničene krivom $y = \ln x$ i pravama $y = 0$, $x = \frac{1}{e}$ i $x = e$.

Deo ravni ograničen krivom $y = \ln x$ i pravama $x = \frac{1}{e}$, $x = e$ i $y = 0$ nalazi se ispod, a deo iznad x-ose. Veličinu površine traženog dela ravni možemo predstaviti kao zbir veličina površina delova ravni na slici označenih sa P_1 i P_2. Neodređeni integral funkcije $y = \ln x$ određujemo metodom parcijalne integracije, dok odgovarajuće određene integrale računamo korišćenjem Njutn–Lajbnicove formule.



$$\int \ln x\,dx = \left\{ \begin{array}{ll} u = \ln x, & dv = dx, \\ du = \frac{dx}{x}, & v = x \end{array} \right\}$$
$$= x\ln x - \int dx = x\ln x - x + C$$

$$P = P_1 + P_2 = -\int_{\frac{1}{e}}^{1} \ln x\,dx + \int_1^e \ln x\,dx$$
$$= -(x\ln x - x)\Big|_{\frac{1}{e}}^{1} + (x\ln x - x)\Big|_1^e$$
$$= 1 - \frac{1}{e} - \frac{1}{e} + e - e + 1 = 2 - \frac{2}{e}$$

Zadatak

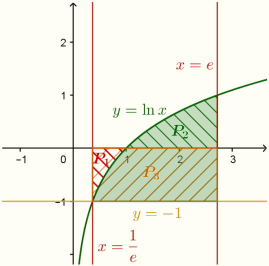
Izračunati veličinu površine ograničene krivom $y = \ln x$ i pravama $y = -1$ i $x = e$.

Ako sa P_3 označimo veličinu površine pravougaonika čija su temena tačke $(\frac{1}{e},0)$, $(e,0)$, $(\frac{1}{e},-1)$ i $(e,-1)$, imamo da je

$$P_3 = \left(e - \frac{1}{e}\right) \cdot 1 = e - \frac{1}{e}.$$

Koristeći prethodni zadatak imamo da je $P = P_2 + P_3 - P_1 = 1 + e - \frac{1}{e} - 1 + \frac{2}{e} = e + \frac{1}{e}$. Direktnim izračunavanjem dobijamo da je veličina tražene površine jednaka

$$\begin{aligned} P &= \int_{\frac{1}{e}}^{e} (\ln x - (-1))\, dx \\ &= (x \ln x - x + x)\Big|_{\frac{1}{e}}^{e} = e + \frac{1}{e}. \end{aligned}$$



Domaći zadatak

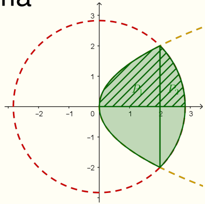
Izračunati veličinu površine ograničene krivom $y = -2\ln(\sqrt{x^2+1}+x)$ i pravama $x = 1$ i $y = 0$. $\left[P = 2\left(\ln(\sqrt{2}+1) - \sqrt{2}+1\right)\right]$

Zadatak

Jednačinama $x^2+y^2=8$ i $x=\frac{1}{2}y^2$ date su kružnica i i parabola. Izračunati veličinu površine njihovog preseka.

Tražena veličina površine jednaka je $P=2(P_1+P_2)$. Tačke preseka kružnice i parabole su rešenja sistema

$$\begin{aligned} x^2+y^2 &= 8 \\ 2x-y^2 &= 0. \end{aligned}$$



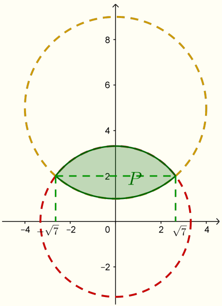
Tražimo koren kvadratne jednačine $x^2+2x-8=0$ za koji važi $x\geq 0$.

Imamo da je $x=2$ i $y=\pm 2$. Veličina površine P_1 jednaka je $\int_0^2 \sqrt{2x}\,dx = \frac{2}{3}\sqrt{2x^3}\Big|_0^2 = \frac{8}{3}$, za P_2 imamo $\int_2^{\sqrt{8}} \sqrt{8-x^2}\,dx =$ $\left(\frac{x}{2}\sqrt{8-x^2}+4\arcsin\frac{x}{\sqrt{8}}\right)\Big|_2^{\sqrt{8}} = 2\pi-2-\pi=\pi-2$, pa je $P=2\pi+\frac{4}{3}$.

Domaći zadatak

Jednačinama $x^2+y^2=11$ i $x^2+(y-5)^2=16$ data su dve kružnice. Izračunati veličinu površine njihovog preseka.

Tačke preseka kružnica su rešenja sistema
$$\begin{aligned} x^2+y^2 &= 11\\ x^2+(y-5)^2 &= 16. \end{aligned}$$
Oduzimanjem druge jednačine od prve dobijamo da je $10y-25=-5$, odnosno da je $y=2$ i $x=\pm\sqrt{7}$. Dalje, imamo da je $P=2\int_0^{\sqrt{7}}\sqrt{11-x^2}-\left(5-\sqrt{16-x^2}\right)dx=$



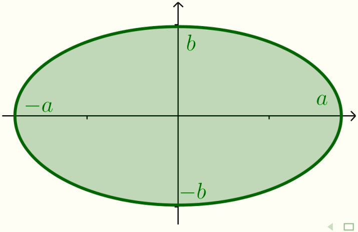
$$2\left(\frac{x}{2}\sqrt{11-x^2}+\frac{11}{2}\arcsin\frac{x}{\sqrt{11}}-5x+\frac{x}{2}\sqrt{16-x^2}+8\arcsin\frac{x}{4}\right)\Bigg|_0^{\sqrt{7}}=$$
$$-5\sqrt{7}+11\arcsin\sqrt{\frac{7}{11}}+16\arcsin\frac{\sqrt{7}}{4}.$$

Zadatak

Izračunati veličinu površine elipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Imamo da je

$$
\begin{aligned}
P &= 4\,\frac{b}{a}\int_0^{\sqrt{a}} \sqrt{a^2 - x^2}\,dx \\
&= 4\,\frac{b}{a}\left(\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}\right)\Bigg|_0^a \\
&= 4\,\frac{b}{a}\,\frac{a^2}{2}\,\frac{\pi}{2} = ab\pi.
\end{aligned}
$$



Domaći zadatak

Izračunati veličinu površine ograničene krivom $y = \frac{e^x}{1+e^x}$ i pravama $y = 0$, $x = 0$ i $x = 1$.

$[\ln(e+1) - \ln(2)]$

Domaći zadatak

Izračunati veličinu površine ograničene krivom $y = \sqrt{x+3}$ i pravama $y = 0$ i $x = 6$.

$\left[18 - 2\sqrt{3}\right]$

Domaći zadatak

Izračunati veličinu površine ograničene krivama $y = \sin x$, $y = \cos x$ i odsečkom x-ose $[0, \frac{\pi}{2}]$.

$\left[2\left(\sqrt{2} - 1\right)\right]$

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