Neodređeni integrali

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Sadržaj

Količnik dva polinoma nazivamo racionalnom funkcijom.

Neka su $P(x)$ i $Q(x)$ polinomi tada je $f(x) = \frac{P(x)}{Q(x)}$ racionalna funkcija čiji je domen $D_f = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$.

Racionalna funkcija je nesvodljiva ako je $NZD(P(x), Q(x)) = 1$, inače je svodljiva. Za svaku svodljivu racionalnu funkciju f postoji nesvodljiva racionalne fukcija g takva da važi:

$$(\forall x \in D_f) f(x) = g(x).$$

$$dgP > dgQ \Rightarrow f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},\ dgR < dgQ$$

$S(x)$ i $R(x)$ su količnik i ostatak pri deljenju polinoma $P(x)$ polinomom $Q(x)$

Racionalnu funkciju $h(x) = \frac{R(x)}{Q(x)}$, $dgR < dgQ$ nazivamo pravom racionalnom funkcijom.

Racionalne funkcije oblika $\frac{A}{(x-a)^k}$ i $\frac{Bx+C}{(x^2+bx+c)^k}$, $b^2-4c<0$, $A,B,C,a,b,c\in\mathbb{R}$, $k\in\mathbb{N}$ nazivaju se parcijalnim razlomcima.

Prava racionalna funkcija $h(x)=\frac{R(x)}{Q(x)}$ može se predstaviti na svom domenu kao zbir parcijalnih razlomaka na sledeći način:

$$h(x)=\sum_{i=1}^{k}\sum_{j=1}^{r_i}\frac{A_{ij}}{(x-x_i)^j}+\sum_{i=1}^{l}\sum_{j=1}^{s_i}\frac{B_{ij}x+C_{ij}}{(x^2+b_ix+c_i)^j},$$

gde je

$$\begin{aligned}Q(x) &= a_n(x-x_1)^{r_1}\cdot(x-x_2)^{r_2}\cdot\ldots\cdot(x-x_k)^{r_k}\cdot\\ &\cdot(x^2+b_1x+c_1)^{s_1}\cdot(x^2+b_2x+c_2)^{s_2}\cdot\ldots\cdot(x^2+b_lx+c_l)^{s_l}\end{aligned}$$

faktorizacija polinoma $Q(x)$ nad poljem realnih brojeva.

$$\int \frac{dx}{x-a} = ln|x-a| + C$$

$$\int \frac{dx}{(x-a)^k} = \frac{1}{1-k} \frac{1}{(x-a)^{k-1}} + C, \quad k \neq 1$$

$$\int \frac{dx}{x^2+px+q} = \int \frac{dx}{(x+\frac{p}{2})^2+\frac{4q-p^2}{4}} = \frac{2}{\sqrt{4q-p^2}} arctg\left(\frac{2x+p}{\sqrt{4q-p^2}}\right) + C$$

$$p^2 - 4q < 0$$

$$x^2 + px + q = x^2 + 2\frac{1}{2}p + \frac{p^2}{4} - \frac{p^2}{4} + q = (x+\frac{p}{2})^2 + \frac{4q-p^2}{4}$$

$$\int \frac{dx}{(x^2+px+q)^k}, \quad k \neq 1$$

Ovaj tip integrala se rešava rekurentnim formulama:

$V_1 = \int \frac{dx}{x^2+px+q}$, integral V_{k+1} dobijamo primenom metode parcijalne integracije na integral V_k.

Zadatak

Naći sledeće integrale:

i) $\int \frac{dx}{x^2-a^2}$, $a \neq 0$,

ii) $\int \frac{x^2dx}{x^2-3x+2}$,

iii) $\int \frac{xdx}{x^3-3x+2}$,

iv) $\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)}$,

v) $\int \frac{xdx}{x^8-1}$,

vi) $\int \frac{dx}{(x^2+x+1)^2}$.

$\int \frac{dx}{x^2-a^2},\ a \neq 0$

$$\begin{aligned}\frac{1}{x^2-a^2} &= \frac{1}{(x-a)(x+a)} &&= \frac{A}{x-a}+\frac{B}{x+a}\\ &= \frac{A(x+a)+B(x-a)}{x^2-a^2} &&= \frac{(A+B)x+(A-B)a}{x^2-a^2}\end{aligned}$$

$$1 = (A+B)x + Aa - Ba$$

$$\begin{aligned}A+B &= 0\\ Aa-Ba &= 1\end{aligned}$$

$$A = \frac{1}{2a},\ B = -\frac{1}{2a}$$

$$\begin{aligned}\int \frac{dx}{x^2-a^2} &= \frac{1}{2a}\left(\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right)\\ &= \frac{1}{2a}(ln|x-a| - ln|x+a|) + C = \frac{1}{2a} ln\left|\frac{x-a}{x+a}\right| + C\end{aligned}$$

$\int \frac{x^2 dx}{x^2-3x+2}$

$$\frac{x^2}{x^2-3x+2} = \frac{x^2-3x+2+3x-2}{x^2-3x+2} = 1 + \frac{3x-2}{x^2-3x+2}$$

$$\frac{3x-2}{x^2-3x+2} = \frac{3x-2}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

$$3x-2 = A(x-2) + B(x-1)$$

$$x = 2 \quad \Rightarrow \quad B = 4$$

$$x = 1 \quad \Rightarrow \quad -A = 1 \Rightarrow A = -1$$

$$\int \frac{x^2 dx}{x^2-3x+2} = \int dx - \int \frac{dx}{x-1} + 4\int \frac{dx}{x-2}$$

$$= x - ln|x-1| + 4ln|x-2| + C$$

$\int \frac{xdx}{x^3-3x+2}$

$$\begin{aligned} x^3-3x+2 &= x(x^2-1)-2(x-1)=(x-1)(x^2+x-2) \\ &= (x-1)(x^2-1+x-1)=(x-1)^2(x+2) \end{aligned}$$

$$\begin{aligned} \frac{x}{x^3-3x+2} &= \frac{x}{(x-1)^2(x+2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+2} \\ &= \frac{A(x-1)(x+2)+B(x+2)+C(x-1)^2}{(x-1)^2(x+2)} \end{aligned}$$

$$x=A(x-1)(x+2)+B(x+2)+C(x-1)^2$$

I način

$$\begin{aligned} x=1 &\Rightarrow 3B=1 &&\Rightarrow B=\tfrac{1}{3} \\ x=-2 &\Rightarrow 9C=-2 &&\Rightarrow C=-\tfrac{2}{9} \\ x=0 &\Rightarrow -2A+2B+C=0 &&\Rightarrow A=\tfrac{2}{9} \end{aligned}$$

$$\int \frac{xdx}{x^3-3x+2}$$

II način

$$\begin{aligned} x &= A(x-1)(x+2)+B(x+2)+C(x-1)^2 \\ &= A(x^2+x-2)+B(x+2)+C(x^2-2x+1) \\ &= (A+C)x^2+(A+B-2C)x+(-2A+2B+C) \end{aligned}$$

$$\begin{aligned} A+C &= 0 \Rightarrow & C &= -A \\ A+B-2C &= 1 \Rightarrow & B &= -A+2C+1=-3A+1 \\ -2A+2B+C &= 0 \Rightarrow & -9A+2 &= 0 \end{aligned}$$

$$A=\frac{2}{9}, \quad B=\frac{1}{3}, \quad C=-\frac{2}{9}$$

$$\begin{aligned} \int \frac{xdx}{x^3-3x+2} &= \frac{2}{9}\int \frac{dx}{(x-1)}+\frac{1}{3}\int \frac{dx}{(x-1)^2}-\frac{2}{9}\int \frac{dx}{(x+2)} \\ &= \frac{2}{9} ln\left|\frac{x-1}{x+2}\right| - \frac{1}{3}\frac{1}{x-1}+C \end{aligned}$$

$\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)}$

$$x^2-4x+4 = (x-2)^2$$

$$x^2-4x+5 = (x-2)^2+1$$

$$\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)} = \int \frac{dx}{(x-2)^2((x-2)^2+1)}$$

$$= \left\{ \begin{array}{c} t=(x-2) \\ dt=dx \end{array} \right\} = \int \frac{dt}{t^2(t^2+1)}$$

$$= \int \frac{t^2+1-t^2}{t^2(t^2+1)} dt = \int \frac{dt}{t^2} - \int \frac{dt}{t^2+1}$$

$$= -\frac{1}{t} - arctgt + C$$

$$= -\frac{1}{x-2} - arctg(x-2) + C$$

$\int \frac{xdx}{x^8-1}$

$$
\begin{aligned}
\int \frac{xdx}{x^8-1} &= \left\{ \begin{array}{c} t = x^2 \\ dt = 2xdx \end{array} \right\} &&= \frac{1}{2}\int \frac{dt}{t^4-1} \\
&= \frac{1}{2}\int \frac{dt}{(t^2-1)(t^2+1)} &&= \frac{1}{4}\int \frac{t^2+1-(t^2-1)}{(t^2-1)(t^2+1)}dt \\
&= \frac{1}{4}\left(\int \frac{dt}{t^2-1} - \int \frac{dt}{t^2+1}\right) &&= \frac{1}{4}\left(\frac{1}{2}ln\left|\frac{t-1}{t+1}\right| - arctgt\right) + C \\
&= \frac{1}{8}ln\left|\frac{x^2-1}{x^2+1}\right| - \frac{1}{4}arctg\, x^2 &&+ \quad C
\end{aligned}
$$

$\int \frac{dx}{(x^2+x+1)^2}$

$$\begin{aligned}
I_1 &= \int \frac{dx}{x^2+x+1} = \left\{ \begin{array}{cc} u = \frac{1}{x^2+x+1} & dv = dx \\ du = -\frac{2x+1}{(x^2+x+1)^2} & v = x \end{array} \right\} \\
&= \frac{x}{x^2+x+1} + \int \frac{2x^2+x}{(x^2+x+1)^2} dx = \frac{x}{x^2+x+1} + \int \frac{2x^2+2x+2-x-2}{(x^2+x+1)^2} dx \\
&= \frac{x}{x^2+x+1} + 2\int \frac{x^2+x+1}{(x^2+x+1)^2} dx - \int \frac{x+2}{(x^2+x+1)^2} dx \\
&= \frac{x}{x^2+x+1} + 2\int \frac{dx}{x^2+x+1} - \frac{1}{2}\int \frac{2x+1+3}{(x^2+x+1)^2} dx \\
&= \frac{x}{x^2+x+1} + 2I_1 - \frac{1}{2}\int \frac{2x+1}{(x^2+x+1)^2} dx - \frac{3}{2}\int \frac{dx}{(x^2+x+1)^2} \\
&= \frac{x}{x^2+x+1} + 2I_1 + \frac{1}{2}\frac{1}{x^2+x+1} - \frac{3}{2} I_2 \\
&= \frac{1}{2}\frac{2x+1}{x^2+x+1} + 2I_1 - \frac{3}{2} I_2
\end{aligned}$$

$\int \frac{dx}{(x^2+x+1)^2}$

$$\int \frac{2x+1}{(x^2+x+1)^2}dx = \left\{ \begin{array}{c} t = x^2+x+1 \\ dt = 2x+1 \end{array} \right\} = \int \frac{dt}{t^2}$$

$$= -\frac{1}{t} + C = -\frac{1}{x^2+x+1} + C$$

$$I_1 = \int \frac{dx}{x^2+x+1} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{2}{\sqrt{3}} arctg \frac{2x+1}{\sqrt{3}} + C$$

$$x^2+x+1 = x^2+2\frac{1}{2}x+\frac{1}{4}+\frac{3}{4} = \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$I_1 = \frac{1}{2}\frac{2x+1}{x^2+x+1} + 2I_1 - \frac{3}{2}I_2$$

$$I_2 = \frac{2}{3}\left(\frac{1}{2}\frac{2x+1}{x^2+x+1} + I_1\right) = \frac{1}{3}\frac{2x+1}{x^2+x+1} + \frac{4}{3\sqrt{3}} arctg \frac{2x+1}{\sqrt{3}} + C$$

Domaći zadatak

Naći sledeće integrale:

i) $\int x ln\frac{1-x}{1+x}dx$, $\left[\frac{1}{2}(x^2+1)ln\frac{1-x}{1+x}-x+C\right]$

ii) $\int \frac{x^3dx}{x^8-2}$, $\left[\frac{\sqrt{2}}{16}ln\left|\frac{x^4-\sqrt{2}}{x^4+\sqrt{2}}\right|+C\right]$

ii) $\int \frac{2^x dx}{1-4^x}$. $\left[\frac{1}{2ln2}ln\left|\frac{2^x+1}{2^x-1}\right|+C\right]$

Domaći zadatak

Naći sledeće integrale:

i) $\int \frac{xdx}{(x+1)(x+2)(x+3)}$, $\left[ln(x+2)^2-\frac{1}{2}ln|x+1|-\frac{3}{2}ln|x+3|+C\right]$

ii) $\int \frac{x^3+1}{x^3-5x^2+6x}dx$. $\left[\frac{1}{6}ln|x|-\frac{9}{2}ln|x-2|+\frac{28}{3}ln|x-3|+x+C\right]$

Domaći zadatak

Naći integral $\int \frac{xdx}{(x-1)^2(x+1)^2}$. $\left[\frac{1}{2(1-x^2)}+C\right]$

Domaći zadatak

Naći sledeće integrale:

$i)\ \int \frac{x^2+5x+4}{x^4+5x^2+4}dx,$ $\left[\frac{5}{6}ln\left|\frac{x^2+1}{x^2+4}\right| + arctgx + C\right]$

$ii)\ \int \frac{x^4 dx}{x^4+10x^2+9},$ $\left[x + \frac{1}{8}arctgx - \frac{27}{8}arctg\frac{x}{3} + C\right]$

$iii)\ \int \frac{x^3-2}{x^3+1}dx,$ $\left[x + ln\frac{\sqrt{x^2-x+1}}{|x+1|} - \sqrt{3}arctg\frac{2x-1}{\sqrt{3}} + C\right]$

$iv)\ \int \frac{xdx}{x^3-1}dx,$ $\left[\frac{1}{3}ln\frac{|x-1|}{\sqrt{x^2+x+1}} + \frac{1}{\sqrt{3}}arctg\frac{2x+1}{\sqrt{3}} + C\right]$

$vi)\ \int \frac{dx}{x(x^2+1)^2}dx,$ $\left[ln\frac{x}{\sqrt{x^2+1}} + \frac{1}{2}\frac{x^2+2}{x^2+1} + C\right]$

$v)\ \int \frac{dx}{x^4+x^2+1}dx.$ $\left[\frac{1}{4}ln\frac{x^2+x+1}{x^2-x+1} + \frac{arctg\frac{2x-1}{\sqrt{3}}}{2\sqrt{3}} + \frac{arctg\frac{2x+1}{\sqrt{3}}}{2\sqrt{3}} + C\right]$

Domaći zadatak

Naći sledeće integrale:

i) $\int \frac{dx}{(x^2+x+2)^2}$, $\left[\frac{2x+1}{7(x^2+x+2)} + \frac{4}{7^{\frac{3}{2}}} arctg\frac{2x+1}{\sqrt{7}} + C\right]$

ii) $\int \frac{x^2dx}{(x^2+4)^2}$, $\left[\frac{1}{4} arctg\frac{x}{2} - \frac{1}{2} fracx x^2 + 4 + C\right]$

iii) $\int \frac{x^3+1}{x(x^2+x+1)^2} dx$. $\left[ln\frac{|x|}{\sqrt{x^2+x+1}} - \frac{arctg\frac{2x+1}{\sqrt{3}}}{3\sqrt{3}} + \frac{3x^2-x+7}{6(x^2+x+1)} + C\right]$

Integrali tipa

$$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_1}{q_1}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_2}{q_2}}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_k}{q_k}}\right) dx,$$

$a, b, c, d \in \mathbb{R}$, $a^2 + b^2 + c^2 + d^2 \neq 0$,
$p_1, p_2, \ldots, p_k \in \mathbb{Z}$, $q_1, q_2, \ldots, q_k \in \mathbb{N}$,
$NZD(p_i, q_i) = 1$ za $1 \leq i \leq k$,
se smenom

$$t^s = \frac{ax+b}{cx+d},\ s = NZS(q_1, q_2, \ldots, q_k),$$

svode na integrale racionalnih funkcija.

Zadatak

Naći sledeće integrale:

i) $\int \frac{dx}{(1+\sqrt{x})}$,

ii) $\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx$,

iii) $\int \frac{1+\sqrt{x+1}}{1-\sqrt[3]{x+1}}dx$,

iv) $\int \sqrt{\frac{x-1}{x+1}}\,dx$,

v) $\int \frac{x^2+\sqrt{x+1}}{\sqrt[3]{(x+1)^2}}dx$.

Domaći zadatak

Naći sledeće integrale:

i) $\int \frac{dx}{x(1+2\sqrt{x}+\sqrt[3]{x})}$,

$\left[lnx - \frac{3}{2}ln(\sqrt[6]{x}+1) - \frac{9}{4}ln(2\sqrt[3]{x}-\sqrt[6]{x}+1) - \frac{3}{2\sqrt{7}}arctg\frac{4\sqrt[6]{x}-1}{\sqrt{7}} + C \right]$

ii) $\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}dx$. $\left[\frac{x^2}{2} - \frac{x\sqrt{x^2-1}}{2} + \frac{1}{2}ln|x+\sqrt{x^2-1}| + C \right]$

$$\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx$$

$$\begin{aligned}
\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx &= \left\{ \begin{array}{cc} x = t^6, & t > 0 \\ dx = 6t^5 dt & \end{array} \right\} \\
&= \int \frac{t^6+t^4+t}{t^6(1+t^2)} 6t^5 dt \\
&= 6\int \frac{t^5+t^3+1}{1+t^2} dt \\
&= 6\int \frac{t^3(t^2+1)+1}{1+t^2} dt \\
&= 6\left(\int t^3 dt + \int \frac{dt}{1+t^2}\right) \\
&= \frac{3}{2}t^4 + 6 arctgt + C \\
&= \frac{3}{2}\sqrt[3]{x^2} + 6 arctg\sqrt[6]{x} + C
\end{aligned}$$

Integrali tipa $\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}}$, gde je $P_n(x)$ polinom n-tog stepena, $n > 1$, $a, b, c \in \mathbb{R}$ i $a \neq 0$ rešavaju se metodom neodređenih koeficijenata:

$$\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}} = Q_{n-1}(x)\sqrt{ax^2+bx+c} + \lambda \int \frac{dx}{\sqrt{ax^2+bx+c}},$$

$Q_{n-1}(x)$ je polinom $n-1$ stepena sa neodređenim koeficijentima i λ je neodređen koeficijent.

Zadatak

Naći sledeće integrale:

i) $\int \frac{x^3dx}{\sqrt{1+2x-x^2}}$,

ii) $\int x^2\sqrt{x^2+4}\,dx$.

Domaći zadatak

Naći sledeće integrale:

i) $\int \frac{x^2+5x+6}{\sqrt{x^2+2x-3}}dx,$

$\left[\frac{x+7}{2}\sqrt{x^2+2x-3}+4ln|x+1+\sqrt{x^2+2x-3}|+C\right]$

ii) $\int \frac{2x^2-6}{\sqrt{x^2+2x+2}}dx,$

$\left[(x-3)\sqrt{x^2+2x+2}-5ln|x+1+\sqrt{x^2+2x+2}|+C\right]$

iii) $\int \frac{x^3-6x^2+11x-6}{\sqrt{x^2+4x+3}}dx,$

$\left[\frac{x^2-14x+111}{3}\sqrt{x^2+4x+3}+66ln|-x-2+\sqrt{x^2+4x+3}|+C\right]$

iv) $\int \frac{3x^3+5}{\sqrt{x^2+4}}dx.$ $\left[(x^2-8)\sqrt{x^2+4}+5ln|x+\sqrt{x^2+4}|+C\right]$

$\int \frac{x^3 dx}{\sqrt{1+2x-x^2}}$

$$\int \frac{x^3 dx}{\sqrt{1+2x-x^2}} = \left(Ax^2+Bx+C\right)\sqrt{1+2x-x^2} + \lambda \int \frac{dx}{\sqrt{1+2x-x^2}}$$

Diferenciranjem prethodne jednakosti dobijamo:

$$\begin{aligned} \frac{x^3}{\sqrt{1+2x-x^2}} &= (2Ax+B)\sqrt{1+2x-x^2} \\ &+ \left(Ax^2+Bx+C\right)\frac{2-2x}{2\sqrt{1+2x-x^2}} + \lambda \frac{1}{\sqrt{1+2x-x^2}} \end{aligned}$$

Množenjem prethodne jednakosti sa $\sqrt{1+2x-x^2}$ dobijamo:

$$x^3 = (2Ax+B)(1+2x-x^2) + (Ax^2+Bx+C)(1-x) + \lambda$$

$$x^3 = -3Ax^3 + (5A-2B)x^2 + (2A+3B-C)x + (B+C+\lambda)$$

$\int \frac{x^3dx}{\sqrt{1+2x-x^2}}$

$$-3A = 1 \quad \Rightarrow \quad A = -\frac{1}{3}$$

$$5A - 2B = 0 \quad \Rightarrow \quad B = -\frac{5}{6}$$

$$2A + 3B - C = 0 \quad \Rightarrow \quad C = -\frac{19}{6}$$

$$B + C + \lambda = 0 \quad \Rightarrow \quad \lambda = 4$$

$$\int \frac{dx}{\sqrt{1+2x-x^2}} = \int \frac{dx}{\sqrt{2-(x-1)^2}} = arcsin\frac{x-1}{\sqrt{2}} + C$$

$$-x^2 + 2x + 1 = -(x^2 - 2x + 1) + 2 = 2 - (x-1)^2$$

$$\begin{aligned} \int \frac{x^3dx}{\sqrt{1+2x-x^2}} &= -\left(\tfrac{1}{3}x^2 + \tfrac{5}{6}x + \tfrac{19}{6}\right)\sqrt{1+2x-x^2} + 4\int \frac{dx}{\sqrt{1+2x-x^2}} \\ &= -\left(\tfrac{1}{3}x^2 + \tfrac{5}{6}x + \tfrac{19}{6}\right)\sqrt{1+2x-x^2} + 4arcsin\frac{x-1}{\sqrt{2}} + C \end{aligned}$$

Integrali tipa

$$\int \frac{Ax+B}{(mx+n)^k\sqrt{ax^2+bx+c}}dx,$$

$A, B, a, b, c, m, n \in \mathbb{R}$, $A^2+B^2 \neq 0$, $m \neq 0$, $a \neq 0$, $k \in \mathbb{N}$ se rešavaju smenom $t = \frac{1}{mx+n}$.

Zadatak

Naći sledeće integrale:

i) $\int \frac{dx}{x^2\sqrt{x^2-x+1}}$, na intervalu od $(0,+\infty)$,

ii) $\int \frac{xdx}{(x+2)^2\sqrt{x^2+2x-5}}$.

Domaći zadatak

Naći sledeće integrale:

i) $\int \frac{dx}{(x-1)^3\sqrt{x^2+3x+1}}$,

ii) $\int \frac{dx}{x\sqrt{x^2+x+1}}$,

iii) $\int \frac{dx}{(x-1)^2\sqrt{1+2x-x^2}}$.

$\int \frac{dx}{x^2\sqrt{x^2-x+1}}$

$$
\begin{aligned}
\int \frac{dx}{x^2\sqrt{x^2-x+1}} &= \left\{ \begin{array}{c} t = \frac{1}{x} \\ dt = \frac{dx}{x^2} \end{array} \right\} = -\int \frac{dt}{\sqrt{\frac{1}{t^2}-\frac{1}{t}+1}} = -\int \frac{tdt}{\sqrt{1-t+t^2}} \\
&= -\frac{1}{2}\int \frac{2t-1+1}{\sqrt{1-t+t^2}}dt = -\frac{1}{2}\left(\int \frac{2t-1}{\sqrt{1-t+t^2}}dt + \int \frac{dt}{\sqrt{1-t+t^2}}\right) \\
&= -\sqrt{1-t+t^2} - \frac{1}{2}ln\left|t-\frac{1}{2}+\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}\right| + C \\
&= -\frac{\sqrt{x^2-x+1}}{x} - \frac{1}{2}ln\left|\frac{2-x+2\sqrt{x^2-x+1}}{2x}\right| + C
\end{aligned}
$$

$\int \frac{dx}{x^2\sqrt{x^2-x+1}}$

$$\frac{1}{2}\int \frac{2t-1}{\sqrt{1-t+t^2}}dt = \left\{ \begin{array}{c} u=\sqrt{1-t+t^2} \\ du=\frac{2t-1}{2\sqrt{1-t+t^2}}dt \end{array} \right\} = \int du$$

$$= u+C=\sqrt{1-t+t^2}+C$$

$$\int \frac{dt}{\sqrt{1-t+t^2}} = \int \frac{dt}{\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}} = ln\left|t-\frac{1}{2}+\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}\right|+C$$

$$t^2-t+1=t^2-2\frac{1}{2}t+\frac{1}{4}+\frac{3}{4}=\left(t-\frac{1}{2}\right)^2+\frac{3}{4}$$

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