

# Neodređeni integrali

## Integracija racionalnih i iracionalnih funkcija

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## 1 Racionalne funkcije

### 1.1 Kratak teorijski pregled

#### 1.1.1 Definicija racionalne funkcije

Količnik dva polinoma nazivamo **racionalnom funkcijom**.

Neka su  $P(x)$  i  $Q(x)$  polinomi tada je  $f(x) = \frac{P(x)}{Q(x)}$  racionalna funkcija čiji je domen  $D_f = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$ .

Racionalna funkcija je **nesvodljiva** ako je  $NZD(P(x), Q(x)) = 1$ , inače je **svodljiva**. Za svaku svodljivu racionalnu funkciju  $f$  postoji nesvodljiva racionalne funkcija  $g$  takva da važi:

$$(\forall x \in D_f) f(x) = g(x).$$

$$dgP > dgQ \Rightarrow f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}, \quad dgR < dgQ$$

$S(x)$  i  $R(x)$  su količnik i ostatak pri deljenju polinoma  $P(x)$  polinomom  $Q(x)$

Racionalnu funkciju  $h(x) = \frac{R(x)}{Q(x)}$ ,  $dgR < dgQ$  nazivamo **pravom racionalnom funkcijom**.

Racionalne funkcije oblika  $\frac{A}{(x-a)^k}$  i  $\frac{Bx+C}{(x^2+bx+c)^k}$ ,  $b^2 - 4c < 0$ ,  $A, B, C, a, b, c \in \mathbb{R}$ ,  $k \in \mathbb{N}$  nazivaju se **parcijalnim razlomcima**.

Prava racionalna funkcija  $h(x) = \frac{R(x)}{Q(x)}$  može se predstaviti na svom domenu kao zbir parcijalnih razlomaka na sledeći način:

$$h(x) = \sum_{i=1}^k \sum_{j=1}^{r_i} \frac{A_{ij}}{(x-x_i)^j} + \sum_{i=1}^l \sum_{j=1}^{s_i} \frac{B_{ij}x + C_{ij}}{(x^2 + b_ix + c_i)^j},$$

gde je

$$Q(x) = a_n(x-x_1)^{r_1} \cdot \dots \cdot (x-x_k)^{r_k} \cdot (x^2 + b_1x + c_1)^{s_1} \cdot \dots \cdot (x^2 + b_lx + c_l)^{s_l}$$

faktorizacija polinoma  $Q(x)$  nad poljem realnih brojeva.

### 1.1.2 Integracija parcijalnih razlomaka

$$\int \frac{dx}{x-a} = \ln|x-a| + C$$

$$\int \frac{dx}{(x-a)^k} = \frac{1}{1-k} \frac{1}{(x-a)^{k-1}} + C, \quad k \neq 1$$

$$\int \frac{dx}{x^2+px+q} = \int \frac{dx}{(x+\frac{p}{2})^2 + \frac{4q-p^2}{4}} = \frac{2}{\sqrt{4q-p^2}} \arctg \left( \frac{2x+p}{\sqrt{4q-p^2}} \right) + C$$

$p^2 - 4q < 0$

$$x^2 + px + q = x^2 + 2\frac{p}{2}x + \frac{p^2}{4} - \frac{p^2}{4} + q = (x + \frac{p}{2})^2 + \frac{4q-p^2}{4}$$

$$\int \frac{dx}{(x^2+px+q)^k}, \quad k \neq 1$$

Ovaj tip integrala se rešava rekurentnim formulama:

$$V_1 = \int \frac{dx}{x^2+px+q}, \quad \text{integral } V_{k+1} \text{ dobijamo primenom metode}$$

parcijalne integracije na integral  $V_k$ .

## 1.2 Zadaci

**Zadatak 1.** Naći sledeće integrale:

i)  $\int \frac{dx}{x^2-a^2}, a \neq 0,$

ii)  $\int \frac{x^2 dx}{x^2-3x+2},$

iii)  $\int \frac{xdx}{x^3-3x+2},$

iv)  $\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)},$

v)  $\int \frac{xdx}{x^8-1},$

vi)  $\int \frac{dx}{(x^2+x+1)^2}.$

$$i) \int \frac{dx}{x^2-a^2}, a \neq 0$$

$$\begin{aligned} \frac{1}{x^2-a^2} &= \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a} \\ &= \frac{A(x+a)+B(x-a)}{x^2-a^2} = \frac{(A+B)x+(A-B)a}{x^2-a^2} \end{aligned}$$

$$1 = (A+B)x + Aa - Ba$$

$$A+B = 0$$

$$Aa - Ba = 1$$

$$A = \frac{1}{2a}, B = -\frac{1}{2a}$$

$$\begin{aligned} \int \frac{dx}{x^2-a^2} &= \frac{1}{2a} \left( \int \frac{dx}{x-a} - \int \frac{dx}{x+a} \right) \\ &= \frac{1}{2a} (\ln|x-a| - \ln|x+a|) + C = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C \end{aligned}$$

$$ii) \int \frac{x^2 dx}{x^2-3x+2}$$

$$\frac{x^2}{x^2-3x+2} = \frac{x^2-3x+2+3x-2}{x^2-3x+2} = 1 + \frac{3x-2}{x^2-3x+2}$$

$$\frac{3x-2}{x^2-3x+2} = \frac{3x-2}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

$$3x-2 = A(x-2) + B(x-1)$$

$$x=2 \Rightarrow B=4$$

$$x=1 \Rightarrow -A=1 \Rightarrow A=-1$$

$$\begin{aligned} \int \frac{x^2 dx}{x^2-3x+2} &= \int dx - \int \frac{dx}{x-1} + 4 \int \frac{dx}{x-2} \\ &= x - \ln|x-1| + 4 \ln|x-2| + C \end{aligned}$$

$$iii) \int \frac{x dx}{x^3-3x+2}$$

$$x^3 - 3x + 2 = x(x^2 - 1) - 2(x - 1) = (x - 1)(x^2 + x - 2)$$

$$= (x - 1)(x^2 - 1 + x - 1) = (x - 1)^2(x + 2)$$

$$\frac{x}{x^3-3x+2} = \frac{x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$= \frac{A(x-1)(x+2)+B(x+2)+C(x-1)^2}{(x-1)^2(x+2)}$$

$$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

I način

$$\begin{aligned}x = 1 &\Rightarrow 3B = 1 &&\Rightarrow B = \frac{1}{3} \\x = -2 &\Rightarrow 9C = -2 &&\Rightarrow C = -\frac{2}{9} \\x = 0 &\Rightarrow -2A + 2B + C = 0 &&\Rightarrow A = \frac{2}{9}\end{aligned}$$

II način

$$\begin{aligned}x &= A(x-1)(x+2) + B(x+2) + C(x-1)^2 \\&= A(x^2 + x - 2) + B(x+2) + C(x^2 - 2x + 1) \\&= (A+C)x^2 + (A+B-2C)x + (-2A+2B+C)\end{aligned}$$

$$\begin{aligned}A+C &= 0 \Rightarrow C = -A \\A+B-2C &= 1 \Rightarrow B = -A+2C+1 = -3A+1 \\-2A+2B+C &= 0 \Rightarrow -9A+2 = 0\end{aligned}$$

$$A = \frac{2}{9}, \quad B = \frac{1}{3}, \quad C = -\frac{2}{9}$$

$$\begin{aligned}\int \frac{x dx}{x^3 - 3x + 2} &= \frac{2}{9} \int \frac{dx}{(x-1)} + \frac{1}{3} \int \frac{dx}{(x-1)^2} - \frac{2}{9} \int \frac{dx}{(x+2)} \\&= \frac{2}{9} \ln \left| \frac{x-1}{x+2} \right| - \frac{1}{3} \frac{1}{x-1} + C\end{aligned}$$

$$iv) \int \frac{dx}{(x^2 - 4x + 4)(x^2 - 4x + 5)}$$

$$x^2 - 4x + 4 = (x-2)^2$$

$$x^2 - 4x + 5 = (x-2)^2 + 1$$

$$\begin{aligned}\int \frac{dx}{(x^2 - 4x + 4)(x^2 - 4x + 5)} &= \int \frac{dx}{(x-2)^2((x-2)^2 + 1)} \\&= \left\{ \begin{array}{l} t = (x-2) \\ dt = dx \end{array} \right\} = \int \frac{dt}{t^2(t^2+1)} \\&= \int \frac{t^2+1-t^2}{t^2(t^2+1)} dt = \int \frac{dt}{t^2} - \int \frac{dt}{t^2+1} \\&= -\frac{1}{t} - \arctg t + C \\&= -\frac{1}{x-2} - \arctg(x-2) + C\end{aligned}$$

$$v) \int \frac{xdx}{x^8-1}$$

$$\begin{aligned} \int \frac{xdx}{x^8-1} &= \left\{ \begin{array}{l} t = x^2 \\ dt = 2xdx \end{array} \right\} = \frac{1}{2} \int \frac{dt}{t^4-1} \\ &= \frac{1}{2} \int \frac{dt}{(t^2-1)(t^2+1)} = \frac{1}{4} \int \frac{t^2+1-(t^2-1)}{(t^2-1)(t^2+1)} dt \\ &= \frac{1}{4} \left( \int \frac{dt}{t^2-1} - \int \frac{dt}{t^2+1} \right) = \frac{1}{4} \left( \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| - \arctg t \right) + C \\ &= \frac{1}{8} \ln \left| \frac{x^2-1}{x^2+1} \right| - \frac{1}{4} \arctg x^2 + C \end{aligned}$$

$$vi) \int \frac{dx}{(x^2+x+1)^2}$$

$$\begin{aligned} I_1 &= \int \frac{dx}{x^2+x+1} = \left\{ \begin{array}{l} u = \frac{1}{x^2+x+1} \quad dv = dx \\ du = -\frac{2x+1}{(x^2+x+1)^2} \quad v = x \end{array} \right\} \\ &= \frac{x}{x^2+x+1} + \int \frac{2x^2+x}{(x^2+x+1)^2} dx = \frac{x}{x^2+x+1} + \int \frac{2x^2+2x+2-x-2}{(x^2+x+1)^2} dx \\ &= \frac{x}{x^2+x+1} + 2 \int \frac{x^2+x+1}{(x^2+x+1)^2} dx - \int \frac{x+2}{(x^2+x+1)^2} dx \\ &= \frac{x}{x^2+x+1} + 2 \int \frac{dx}{x^2+x+1} - \frac{1}{2} \int \frac{2x+1+3}{(x^2+x+1)^2} dx \\ &= \frac{x}{x^2+x+1} + 2I_1 - \frac{1}{2} \int \frac{2x+1}{(x^2+x+1)^2} dx - \frac{3}{2} \int \frac{dx}{(x^2+x+1)^2} \\ &= \frac{x}{x^2+x+1} + 2I_1 + \frac{1}{2} \frac{1}{x^2+x+1} - \frac{3}{2} I_2 \\ &= \frac{1}{2} \frac{2x+1}{x^2+x+1} + 2I_1 - \frac{3}{2} I_2 \end{aligned}$$

$$\begin{aligned} \int \frac{2x+1}{(x^2+x+1)^2} dx &= \left\{ \begin{array}{l} t = x^2+x+1 \\ dt = 2x+1 \end{array} \right\} = \int \frac{dt}{t^2} \\ &= -\frac{1}{t} + C = -\frac{1}{x^2+x+1} + C \end{aligned}$$

$$I_1 = \int \frac{dx}{x^2+x+1} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{2}{\sqrt{3}} \arctg \frac{2x+1}{\sqrt{3}} + C$$

$$x^2+x+1 = x^2 + 2\frac{1}{2}x + \frac{1}{4} + \frac{3}{4} = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$I_1 = \frac{1}{2} \frac{2x+1}{x^2+x+1} + 2I_1 - \frac{3}{2} I_2$$

$$I_2 = \frac{2}{3} \left( \frac{1}{2} \frac{2x+1}{x^2+x+1} + I_1 \right) = \frac{1}{3} \frac{2x+1}{x^2+x+1} + \frac{4}{3\sqrt{3}} \arctg \frac{2x+1}{\sqrt{3}} + C$$

### 1.3 Domaći zadaci

**Domaći zadatak 2.** Naći sledeće integrale:

$$i) \int x \ln \frac{1-x}{1+x} dx, \quad \left[ \frac{1}{2} (x^2 + 1) \ln \frac{1-x}{1+x} - x + C \right]$$

$$ii) \int \frac{x^3 dx}{x^8 - 2}, \quad \left[ \frac{\sqrt{2}}{16} \ln \left| \frac{x^4 - \sqrt{2}}{x^4 + \sqrt{2}} \right| + C \right]$$

$$ii) \int \frac{2^x dx}{1-4^x}. \quad \left[ \frac{1}{2 \ln 2} \ln \left| \frac{2^x + 1}{2^x - 1} \right| + C \right]$$

**Domaći zadatak 3.** Naći sledeće integrale:

$$i) \int \frac{x dx}{(x+1)(x+2)(x+3)}, \quad \left[ \ln(x+2)^2 - \frac{1}{2} \ln|x+1| - \frac{3}{2} \ln|x+3| + C \right]$$

$$ii) \int \frac{x^3 + 1}{x^3 - 5x^2 + 6x} dx. \quad \left[ \frac{1}{6} \ln|x| - \frac{9}{2} \ln|x-2| + \frac{28}{3} \ln|x-3| + x + C \right]$$

**Domaći zadatak 4.** Naći integral  $\int \frac{x dx}{(x-1)^2(x+1)^2}. \quad \left[ \frac{1}{2(1-x^2)} + C \right]$

**Domaći zadatak 5.** Naći sledeće integrale:

$$i) \int \frac{x^2 + 5x + 4}{x^4 + 5x^2 + 4} dx, \quad \left[ \frac{5}{6} \ln \left| \frac{x^2 + 1}{x^2 + 4} \right| + \arctg x + C \right]$$

$$ii) \int \frac{x^4 dx}{x^4 + 10x^2 + 9}, \quad \left[ x + \frac{1}{8} \arctg x - \frac{27}{8} \arctg \frac{x}{3} + C \right]$$

$$iii) \int \frac{x^3 - 2}{x^3 + 1} dx, \quad \left[ x + \ln \frac{\sqrt{x^2 - x + 1}}{|x+1|} - \sqrt{3} \arctg \frac{2x-1}{\sqrt{3}} + C \right]$$

$$iv) \int \frac{x dx}{x^3 - 1}, \quad \left[ \frac{1}{3} \ln \frac{|x-1|}{\sqrt{x^2 + x + 1}} + \frac{1}{\sqrt{3}} \arctg \frac{2x+1}{\sqrt{3}} + C \right]$$

$$vi) \int \frac{dx}{x(x^2+1)^2}, \quad \left[ \ln \frac{x}{\sqrt{x^2+1}} + \frac{1}{2} \frac{x^2+2}{x^2+1} + C \right]$$

$$v) \int \frac{dx}{x^4 + x^2 + 1}. \quad \left[ \frac{1}{4} \ln \frac{x^2 + x + 1}{x^2 - x + 1} + \frac{\arctg \frac{2x-1}{\sqrt{3}}}{2\sqrt{3}} + \frac{\arctg \frac{2x+1}{\sqrt{3}}}{2\sqrt{3}} + C \right]$$

**Domaći zadatak 6.** Naći sledeće integrale:

$$i) \int \frac{dx}{(x^2 + x + 2)^2}, \quad \left[ \frac{2x+1}{7(x^2+x+2)} + \frac{4}{7^{\frac{3}{2}}} \arctg \frac{2x+1}{\sqrt{7}} + C \right]$$

$$ii) \int \frac{x^2 dx}{(x^2 + 4)^2}, \quad \left[ \frac{1}{4} \arctg \frac{x}{2} - \frac{1}{2} \frac{1}{x^2 + 4} + C \right]$$

$$iii) \int \frac{x^3 + 1}{x(x^2 + x + 1)^2} dx. \quad \left[ \ln \frac{|x|}{\sqrt{x^2 + x + 1}} - \frac{\arctg \frac{2x+1}{\sqrt{3}}}{3\sqrt{3}} + \frac{3x^2 - x + 7}{6(x^2 + x + 1)} + C \right]$$

## 2 Iracionalne funkcije

### 2.1 Integrali tipa $\int R(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_1}{q_1}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_2}{q_2}}, \dots, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_k}{q_k}}) dx$

Integrali tipa

$$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_1}{q_1}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_2}{q_2}}, \dots, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_k}{q_k}}\right) dx,$$

$$a, b, c, d \in \mathbb{R}, a^2 + b^2 + c^2 + d^2 \neq 0,$$

$$p_1, p_2, \dots, p_k \in \mathbb{Z}, q_1, q_2, \dots, q_k \in \mathbb{N},$$

$$NZD(p_i, q_i) = 1 \text{ za } 1 \leq i \leq k,$$

se smenom

$$t^s = \frac{ax+b}{cx+d}, s = NZS(q_1, q_2, \dots, q_k),$$

svode na integrale racionalnih funkcija.

**Zadatak 7.** Naći sledeće integrale:

i)  $\int \frac{dx}{(1+\sqrt{x})},$

ii)  $\int \frac{x + \sqrt[3]{x^2} + \sqrt{x}}{x(1 + \sqrt[3]{x})} dx,$

iii)  $\int \frac{1 + \sqrt{x+1}}{1 - \sqrt[3]{x+1}} dx,$

iv)  $\int \sqrt{\frac{x-1}{x+1}} dx,$

v)  $\int \frac{x^2 + \sqrt{x+1}}{\sqrt[3]{(x+1)^2}} dx.$

$$\begin{aligned} \int \frac{x + \sqrt[3]{x^2} + \sqrt{x}}{x(1 + \sqrt[3]{x})} dx &= \left\{ \begin{array}{l} x = t^6, \quad t > 0 \\ dx = 6t^5 dt \end{array} \right\} \\ &= \int \frac{t^6 + t^4 + t}{t^6(1+t^2)} 6t^5 dt \\ &= 6 \int \frac{t^5 + t^3 + 1}{1+t^2} dt \\ &= 6 \int \frac{t^3(t^2+1)+1}{1+t^2} dt \\ &= 6 \left( \int t^3 dt + \int \frac{dt}{1+t^2} \right) \\ &= \frac{3}{2} t^4 + 6 \arctg t + C \\ &= \frac{3}{2} \sqrt[3]{x^2} + 6 \arctg \sqrt[6]{x} + C \end{aligned}$$

**Domaći zadatak 8.** Naći sledeće integrale:

i)  $\int \frac{dx}{x(1+2\sqrt{x}+\sqrt[3]{x})}, \left[ \ln x - \frac{3}{2} \ln(\sqrt[6]{x}+1) - \frac{9}{4} \ln(2\sqrt[3]{x} - \sqrt[6]{x}+1) - \frac{3}{2\sqrt{7}} \arctg \frac{4\sqrt[6]{x}-1}{\sqrt{7}} + C \right]$

ii)  $\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} dx. \left[ \frac{x^2}{2} - \frac{x\sqrt{x^2-1}}{2} + \frac{1}{2} \ln|x + \sqrt{x^2-1}| + C \right]$

## 2.2 Integrali tipa $\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}}$

Integrali tipa  $\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}}$ , gde je  $P_n(x)$  polinom  $n$ -tog stepena,  $n > 1$ ,  $a, b, c \in \mathbb{R}$  i  $a \neq 0$  rešavaju se metodom neodređenih koeficijenata:

$$\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}} = Q_{n-1}(x)\sqrt{ax^2+bx+c} + \lambda \int \frac{dx}{\sqrt{ax^2+bx+c}},$$

$Q_{n-1}(x)$  je polinom  $n-1$  stepena sa neodređenim koeficijentima i  $\lambda$  je neodređen koeficijent.

**Zadatak 9.** Naći sledeće integrale:

i)  $\int \frac{x^3 dx}{\sqrt{1+2x-x^2}},$

ii)  $\int x^2 \sqrt{x^2+4} dx.$

$$\int \frac{x^3 dx}{\sqrt{1+2x-x^2}} = (Ax^2+Bx+C)\sqrt{1+2x-x^2} + \lambda \int \frac{dx}{\sqrt{1+2x-x^2}}$$

Diferenciranjem prethodne jednakosti dobijamo:

$$\begin{aligned} \frac{x^3}{\sqrt{1+2x-x^2}} &= (2Ax+B)\sqrt{1+2x-x^2} \\ &+ (Ax^2+Bx+C)\frac{2-2x}{2\sqrt{1+2x-x^2}} + \lambda \frac{1}{\sqrt{1+2x-x^2}} \end{aligned}$$

Množenjem prethodne jednakosti sa  $\sqrt{1+2x-x^2}$  dobijamo:

$$\begin{aligned} x^3 &= (2Ax+B)(1+2x-x^2) + (Ax^2+Bx+C)(1-x) + \lambda \\ x^3 &= -3Ax^3 + (5A-2B)x^2 + (2A+3B-C)x + (B+C+\lambda) \end{aligned}$$

$$-3A = 1 \Rightarrow A = -\frac{1}{3}$$

$$5A - 2B = 0 \Rightarrow B = -\frac{5}{6}$$

$$2A + 3B - C = 0 \Rightarrow C = -\frac{19}{6}$$

$$B + C + \lambda = 0 \Rightarrow \lambda = 4$$

$$\int \frac{dx}{\sqrt{1+2x-x^2}} = \int \frac{dx}{\sqrt{2-(x-1)^2}} = \arcsin \frac{x-1}{\sqrt{2}} + C$$

$$-x^2 + 2x + 1 = -(x^2 - 2x + 1) + 2 = 2 - (x-1)^2$$

$$\begin{aligned} \int \frac{x^3 dx}{\sqrt{1+2x-x^2}} &= -\left(\frac{1}{3}x^2 + \frac{5}{6}x + \frac{19}{6}\right)\sqrt{1+2x-x^2} + 4 \int \frac{dx}{\sqrt{1+2x-x^2}} \\ &= -\left(\frac{1}{3}x^2 + \frac{5}{6}x + \frac{19}{6}\right)\sqrt{1+2x-x^2} + 4\arcsin \frac{x-1}{\sqrt{2}} + C \end{aligned}$$



**Domaći zadatak 10.** Naći sledeće integrale:

$$\begin{aligned}
 \text{i)} \int \frac{x^2+5x+6}{\sqrt{x^2+2x-3}} dx, & \quad \left[ \frac{x+7}{2} \sqrt{x^2+2x-3} + 4 \ln|x+1+\sqrt{x^2+2x-3}| + C \right] \\
 \text{ii)} \int \frac{2x^2-6}{\sqrt{x^2+2x+2}} dx, & \quad \left[ (x-3)\sqrt{x^2+2x+2} - 5 \ln|x+1+\sqrt{x^2+2x+2}| + C \right] \\
 \text{iii)} \int \frac{x^3-6x^2+11x-6}{\sqrt{x^2+4x+3}} dx, & \quad \left[ \frac{x^2-14x+11}{3} \sqrt{x^2+4x+3} + 66 \ln|-x-2+\sqrt{x^2+4x+3}| + C \right] \\
 \text{iv)} \int \frac{3x^3+5}{\sqrt{x^2+4}} dx. & \quad \left[ (x^2-8)\sqrt{x^2+4} + 5 \ln|x+\sqrt{x^2+4}| + C \right]
 \end{aligned}$$

### 2.3 Integrali tipa $\int \frac{Ax+B}{(mx+n)^k \sqrt{ax^2+bx+c}} dx$

Integrali tipa

$$\int \frac{Ax+B}{(mx+n)^k \sqrt{ax^2+bx+c}} dx,$$

$A, B, a, b, c, m, n \in \mathbb{R}, A^2+B^2 \neq 0, m \neq 0, a \neq 0, k \in \mathbb{N}$  se rešavaju smenom  $t = \frac{1}{mx+n}$ .

**Zadatak 11.** Naći sledeće integrale:

$$\text{i)} \int \frac{dx}{x^2 \sqrt{x^2-x+1}}, \text{ na intervalu od } (0, +\infty),$$

$$\text{ii)} \int \frac{xdx}{(x+2)^2 \sqrt{x^2+2x-5}}.$$

$$\begin{aligned}
 \int \frac{dx}{x^2 \sqrt{x^2-x+1}} &= \left\{ \begin{array}{l} t = \frac{1}{x} \\ dt = -\frac{dx}{x^2} \end{array} \right\} = - \int \frac{dt}{\sqrt{\frac{1}{t^2} - \frac{1}{t} + 1}} = - \int \frac{tdt}{\sqrt{1-t+t^2}} \\
 &= -\frac{1}{2} \int \frac{2t-1+1}{\sqrt{1-t+t^2}} dt = -\frac{1}{2} \left( \int \frac{2t-1}{\sqrt{1-t+t^2}} dt + \int \frac{dt}{\sqrt{1-t+t^2}} \right) \\
 &= -\sqrt{1-t+t^2} - \frac{1}{2} \ln \left| t - \frac{1}{2} + \sqrt{\left(t - \frac{1}{2}\right)^2 + \frac{3}{4}} \right| + C \\
 &= -\frac{\sqrt{x^2-x+1}}{x} - \frac{1}{2} \ln \left| \frac{2-x+2\sqrt{x^2-x+1}}{2x} \right| + C
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{2} \int \frac{2t-1}{\sqrt{1-t+t^2}} dt &= \left\{ \begin{array}{l} u = \sqrt{1-t+t^2} \\ du = \frac{2t-1}{2\sqrt{1-t+t^2}} dt \end{array} \right\} = \int du \\
 &= u + C = \sqrt{1-t+t^2} + C
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{dt}{\sqrt{1-t+t^2}} &= \int \frac{dt}{\sqrt{\left(t - \frac{1}{2}\right)^2 + \frac{3}{4}}} = \ln \left| t - \frac{1}{2} + \sqrt{\left(t - \frac{1}{2}\right)^2 + \frac{3}{4}} \right| + C \\
 t^2 - t + 1 &= t^2 - 2\frac{1}{2}t + \frac{1}{4} + \frac{3}{4} = \left(t - \frac{1}{2}\right)^2 + \frac{3}{4}
 \end{aligned}$$

**Domaći zadatak 12.** Naći sledeće integrale:

$$\text{i)} \int \frac{dx}{(x-1)^3 \sqrt{x^2+3x+1}},$$

$$ii) \int \frac{dx}{x\sqrt{x^2+x+1}},$$

$$iii) \int \frac{dx}{(x-1)^2\sqrt{1+2x-x^2}}.$$

### 3 Literatura

#### Literatura

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