FIZIKA SI

Časovi 7 i 8

Dinamika rotacije i statika

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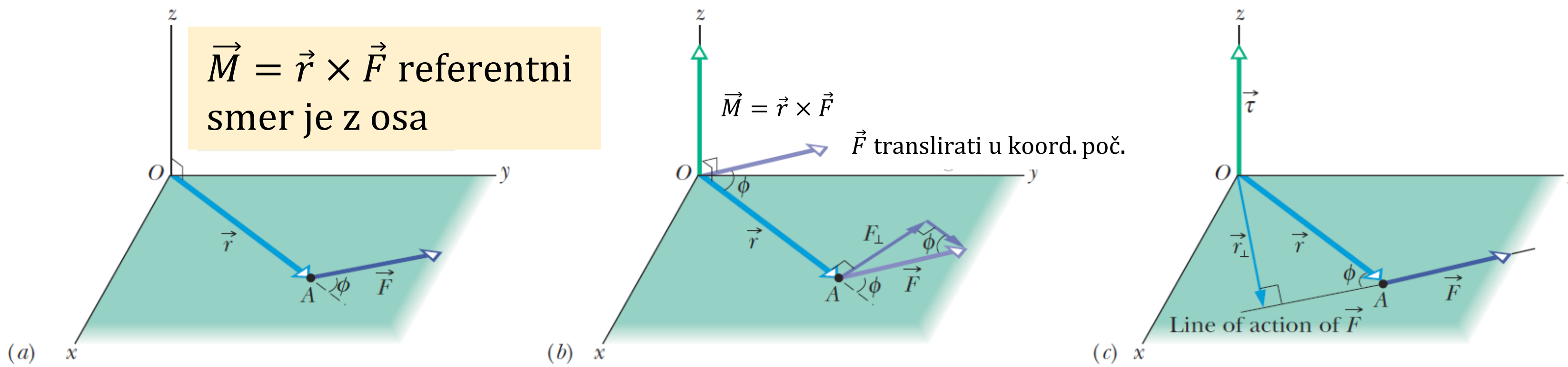
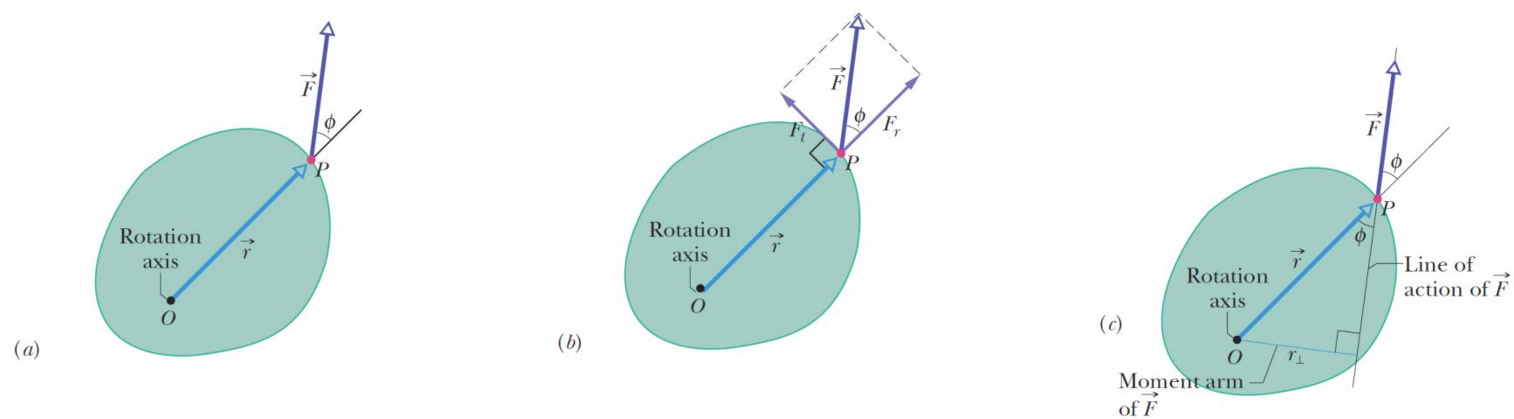
Pojam momenta sile

$\vec{M} = \vec{r} \times \vec{F}$ referentni smer je z osa



Samo tangencijalna komponenta dovodi do rotacije



$\vec{M} = \vec{r} \times \vec{F}$

$\vec{M} = rF_\tau \vec{e}_z = r(F\sin\phi)\vec{e}_z$

$$\vec{M} = r_\perp F \vec{e}_z = (r\sin\phi)F\vec{e}_z$$

II Njutnov zakon za rotaciju MT

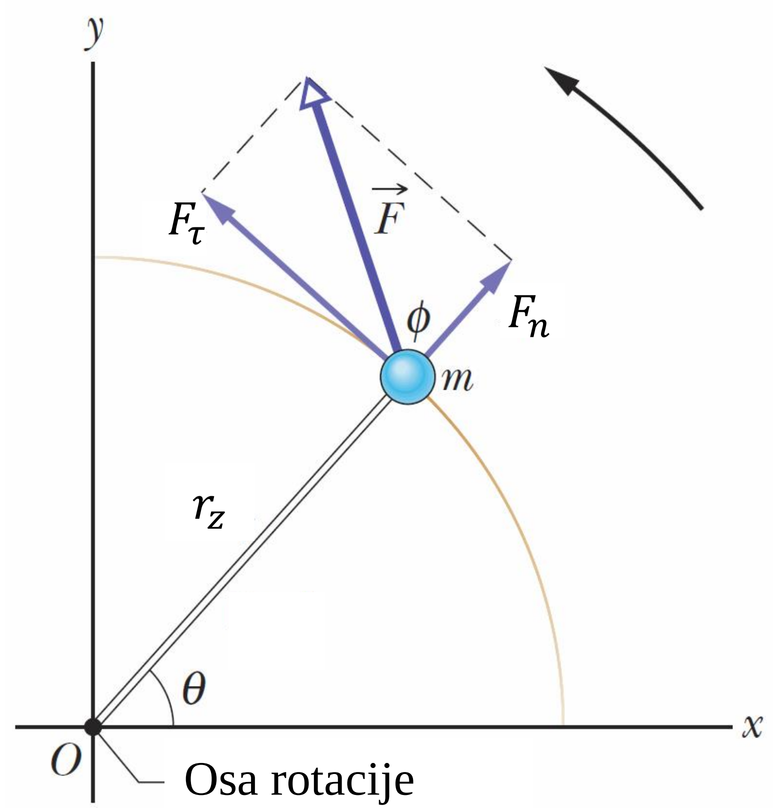
Referentni smer za $\vec{\omega}, \vec{\alpha}, \vec{M}$ je smer z ose

$$F_\tau = ma_\tau = m(\alpha r_z)$$

$$\begin{aligned}\vec{M} &= \vec{r} \times \vec{F} = r_z F_\tau \vec{e}_z \\ &= r_z(m\alpha r_z)\vec{e}_z \\ &= (mr_z^2)\alpha\vec{e}_z \\ &= I_z \alpha \vec{e}_z = I_z \vec{\alpha}\end{aligned}$$

$I_z = mr_z^2$ – moment inercije MT u odnosu na fiksnu z osu

$$\vec{M} = \vec{r} \times \vec{F} = I\vec{\alpha}$$



Kada na MT deluje više eksternih sila $\sum_i \vec{F}_i = \vec{F}_{rez}$ primenom superpozicije

$$\vec{M}_{rez} = \vec{r} \times \vec{F}_{rez} = I\vec{\alpha}$$

Moment količine kretanja MT i njegova promena (II NJ.Z. alternativno)

- Polazeći od II Njutnovog zakona za translaciju MT mase m $m\frac{\mathrm{d}\vec{v}}{dt} = \vec{F}$ i vektorskim množenjem vektorom položaja u odnosu na nepokretnu tačku O daje:

$$\vec{r} \times \left(m\frac{\mathrm{d}\vec{v}}{dt} \right) = \vec{r} \times \vec{F} = \vec{M}_O(\vec{F}).$$

- Na osnovu:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{r} \times m\vec{v}) = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times (m\vec{v}) + \vec{r} \times \left(m\frac{\mathrm{d}\vec{v}}{dt} \right) = \underbrace{\vec{v} \times (m\vec{v})}_{0} + \vec{r} \times \left(m\frac{\mathrm{d}\vec{v}}{dt} \right)$$

- sledi $\frac{\mathrm{d}}{\mathrm{d}t}(\vec{r} \times m\vec{v}) = \vec{r} \times \left(m\frac{\mathrm{d}\vec{v}}{dt} \right)$, te je:

$$\frac{\mathrm{d}\vec{L}_O}{\mathrm{d}t} = \vec{M}_O(\vec{F}),$$

$$\vec{L}_O = \vec{r} \times m\vec{v}$$

moment količine kretanja u odnosu na O

$$\vec{L}_O = I\vec{\omega}$$

rotacija oko O

Rad, snaga i kinetička energija pri rotaciji MT oko fiksne ose

- Elementarni rad rezultantne spoljašnje sile $\vec{F}$ pri rotaciji MT (pređeni put $r\mathrm{d}\theta$):

$$\mathrm{d}A = \vec{F}\mathrm{d}\vec{r} = F_\tau r\mathrm{d}\theta = Md\theta,$$

- gde su F_τ i M algebarske vrednosti tangencijalne sile i momenta.
- Rad pri ugaonom pomaku od θ_i do θ_f je:

$$A = \int_{\theta_i}^{\theta_f} \mathrm{d}A = \int_{\theta_i}^{\theta_f} M\mathrm{d}\theta\,.$$

- Brzina kojom sila vrši rad (snaga sile F) je:

$$P = \frac{\mathrm{d}A}{\mathrm{d}t} = M\frac{\mathrm{d}\theta}{\mathrm{d}t} = M\omega.$$

- Rad koji izvrši rezultantna eksterna sila je:

$$A_{if} = \Delta E_k = E_{k,f} - E_{k,i} = \frac{1}{2}m\left(v_f^2 - v_i^2\right) = \frac{1}{2}m\left(r^2\omega_f^2 - r^2\omega_i^2\right)$$

$$= \frac{1}{2}(mr^2)\left(\omega_f^2 - \omega_i^2\right) = \frac{1}{2}I\left(\omega_f^2 - \omega_i^2\right).$$

- Kinetička energija je:

$$E_{k,rot} = \frac{1}{2}I\omega^2.$$

Rotacija krutog tela i sistema MT: Pojam momenta inercije

- Posmatrajmo čoveka koji drži dva tega i vrti se na stolici.
- Uočava se da mu se ugaona brzina rotacije menja pri širenju i skupljanju ruku, iako mu centar mase ostaje nepromenljiv (kasnije ćemo detaljnije objasniti kroz zakon održanja momenta količine kretanja).
- Osobina sistema koji rotira zavisi od raspodele mase u odnosu na osu rotacije (a ne samo od položaja CM kao kod translacije).

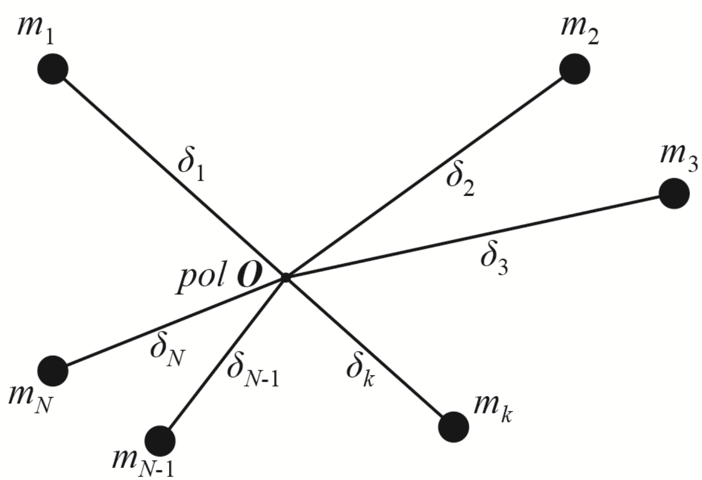
Moment inercije sistema MT i krutog tela u odnosu na pol *O*

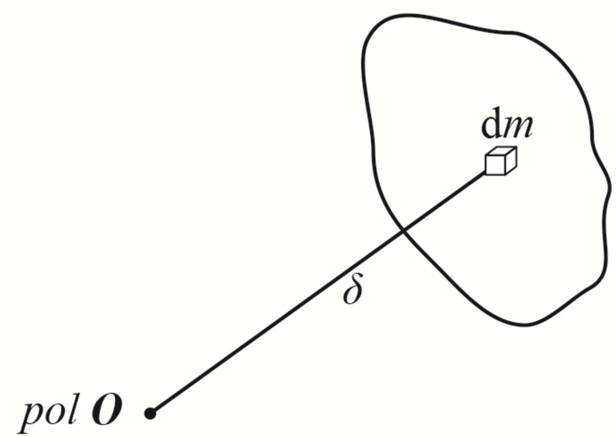
- Sistem MT:

$$I_O = \sum_{k=1}^{N} m_k \delta_k^2$$

- Kruto telo:

$$I_O = \iiint_m \delta^2 \mathrm{d}m = \iiint_V \delta^2 \rho \mathrm{d}V$$





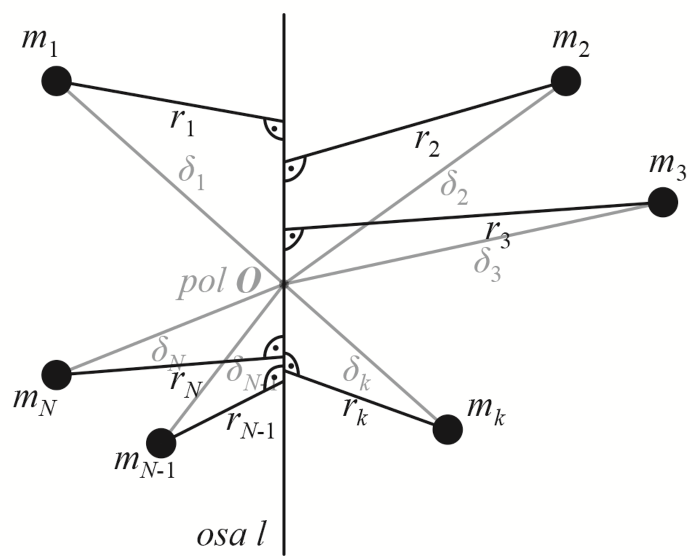
Moment inercije prema proizvoljnoj osi OO' (osa l)

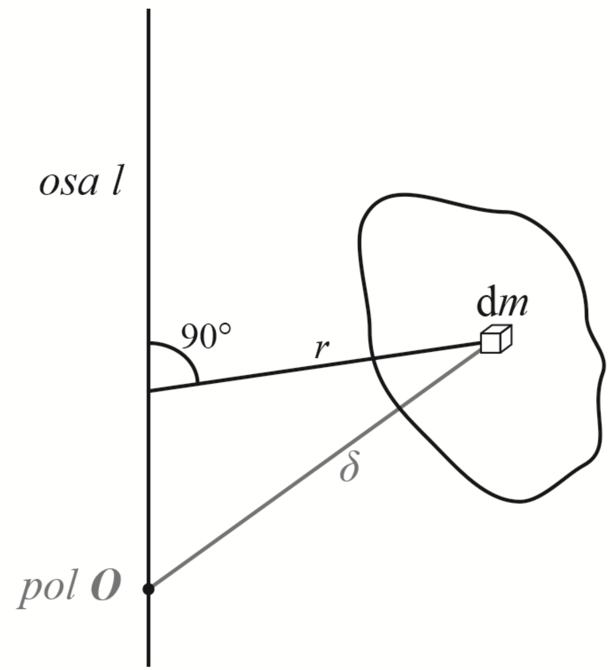
- Sistema MT

$$I_{OO'} = \sum_{k=1}^{N} m_k r_k^2$$

- Krutog tela

$$I_{OO'} = \iiint_m r^2 \mathrm{d}m = \iiint_V r^2 \rho \mathrm{d}V$$





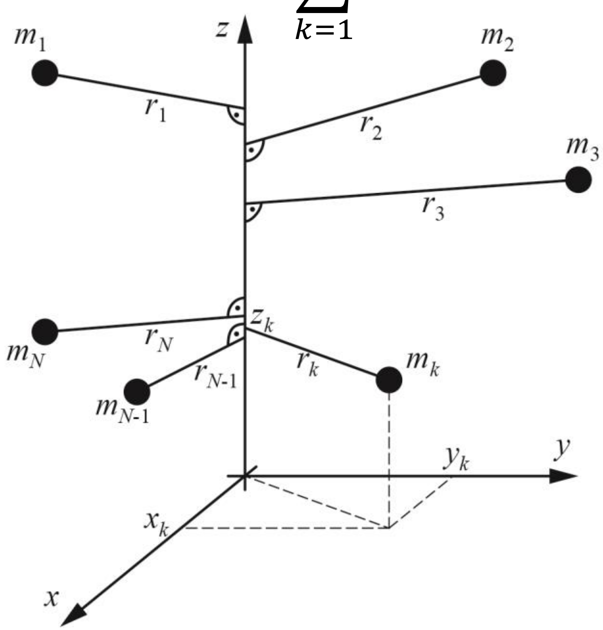
Moment inercije prema koordinatnim osama

- Sistema MT:

$$I_x = \sum_{k=1}^{N} m_k\left(y_k^2 + z_k^2\right)$$

$$I_y = \sum_{k=1}^{N} m_k\left(x_k^2 + z_k^2\right)$$

$$I_z = \sum_{k=1}^{N} m_k\left(x_k^2 + y_k^2\right)$$

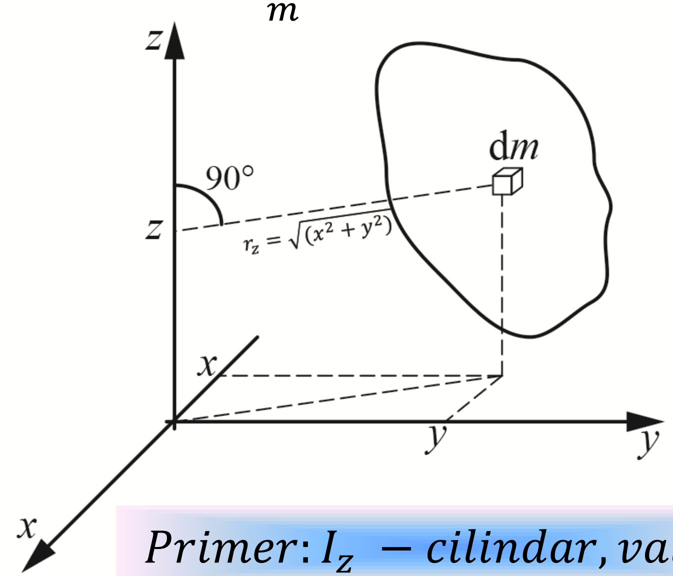


- Krutog tela:

$$I_x = \int_m \left(y^2 + z^2\right)\mathrm{d}m$$

$$I_y = \int_m \left(x^2 + z^2\right)\mathrm{d}m$$

$$I_z = \int_m \left(x^2 + y^2\right)\mathrm{d}m$$



$Primer: I_z \ - cilindar, valjak$

Moment inercije prema koordinatnom početku

- Sistema MT:

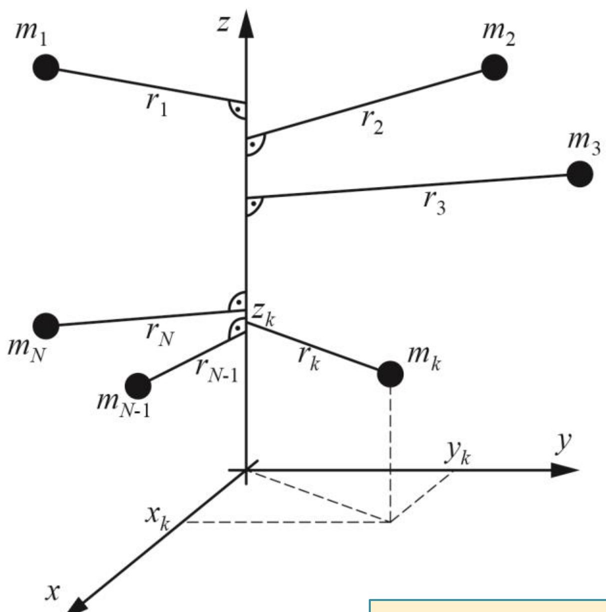
$$I_O = \sum_{k=1}^{N} m_k \delta_k^2 = \sum_{k=1}^{N} m_k (x_k^2 + y_k^2 + z_k^2)$$

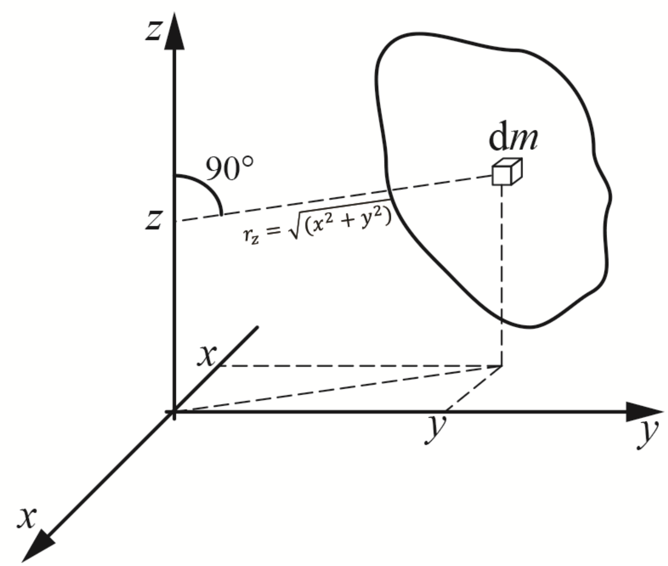
$$I_x + I_y + I_z = \sum_{k=1}^{N} 2m_k (x_k^2 + y_k^2 + z_k^2)$$

- Krutog tela:

$$I_O = \int_m \delta^2 \mathrm{d}m = \int_m (x^2 + y^2 + z^2)\mathrm{d}m$$

$$I_x + I_y + I_z = 2\int_m (x^2 + y^2 + z^2)\mathrm{d}m$$





$$I_x + I_y + I_z = 2I_O$$

Primer: I_{lopte}

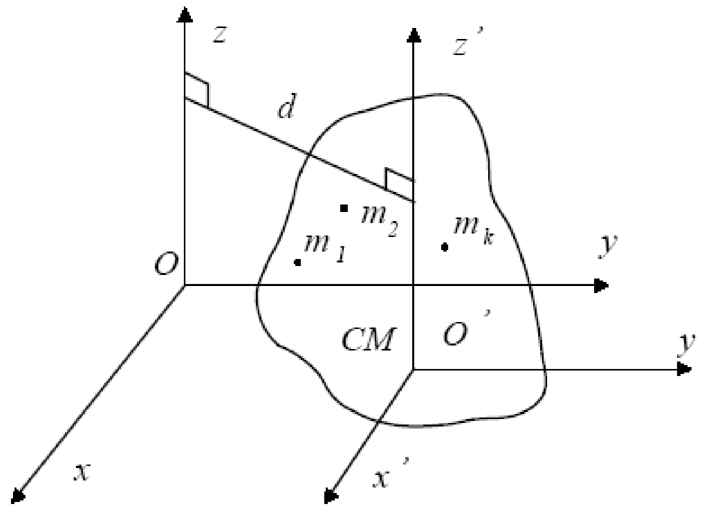
Teorema o normalnim osama

- Posmatra se rotacija tankog pločastog tela oko ose normalne na njegovu površ
- Tada je $I_O = I_z$, pa je:
- $2I_z = I_x + I_y + I_z$

$$I_z = I_x + I_y$$

$Primer: I_x\ poludiska$

Teorema o paralelnim osama: Štajnerova teorema



$$x_C^2 + y_C^2 = d^2$$

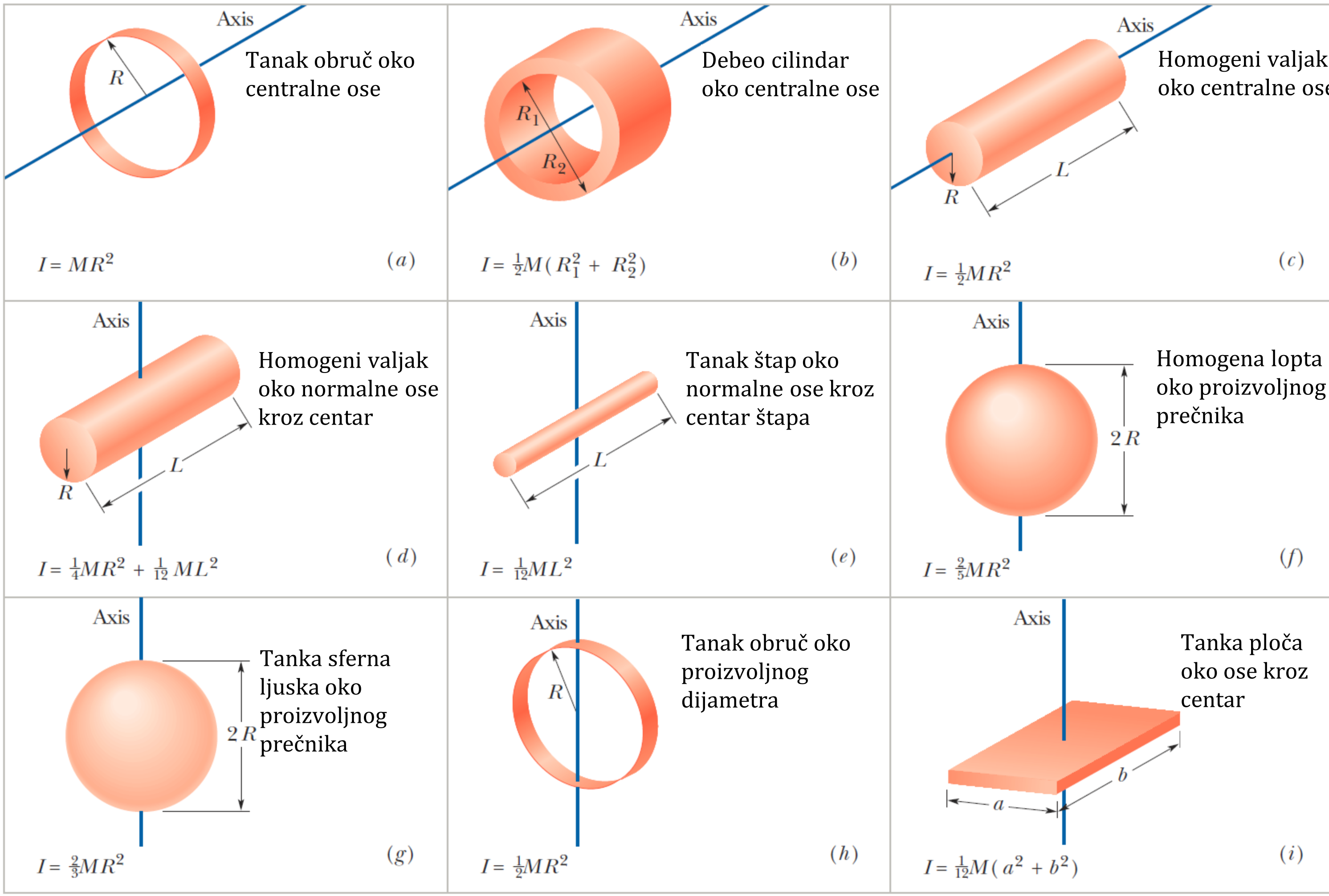
$$I_{Oz} = \sum_k m_k[(x_k' + x_C)^2 + (y_k' + y_C)^2]$$

$$= \sum_k m_k(x_k'^2 + y_k'^2) + 2x_C \underbrace{\sum_k m_k x_k'}_{0} + 2y_C \underbrace{\sum_k m_k y_k'}_{0} + (x_C^2 + y_C^2)\sum_k m_k \, .$$

$$I_{Oz} = I_{O'z'} + md^2 = I_{CM} + md^2$$

$Primer: I_{štapa}$

Moment inercije određenih tela

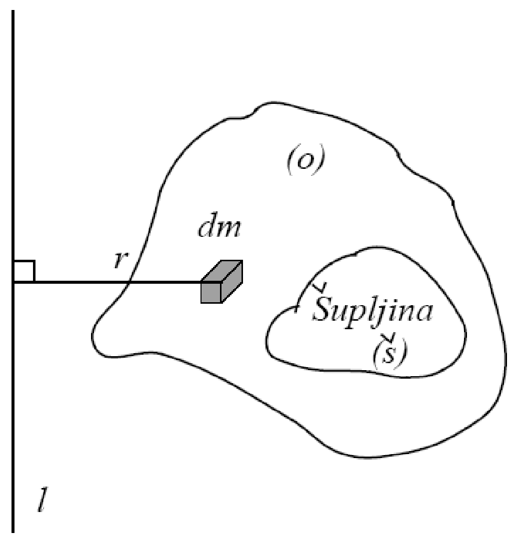


Princip superpozicije pri određivanju momenta inercije

- Ako iz tela zapremine V isečemo deo tako da ostane šupljina $V_{š}$:

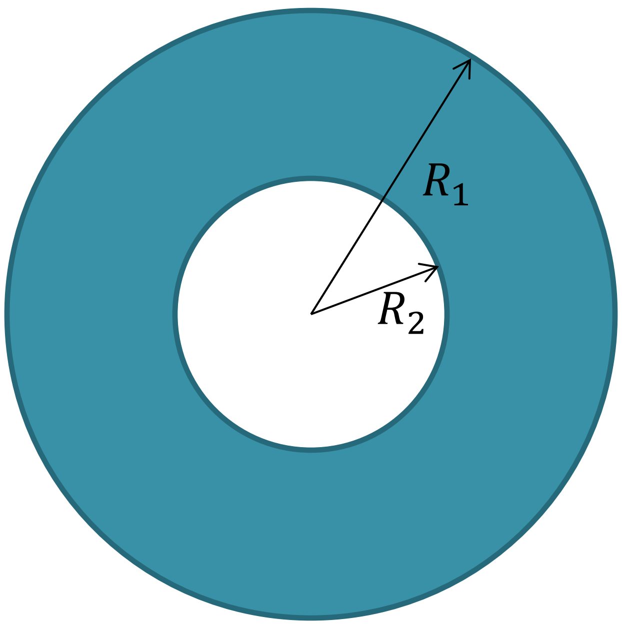
$$I_l = \int_V r^2 \rho dV = \int_{V_0} r^2 \rho dV + \int_{V_š} r^2 \rho dV = I_{lO} + I_{lš}$$

$$V = V_O + V_š$$



$$I_{lO} = I_l - I_š$$

Princip superpozicije pri određivanju momenta inercije: primer



- Masa prstena je m.
- Površinska gustina:

$$\rho_S = \frac{m}{\pi(R_1^2 - R_2^2)}$$

- Masa celog diska i isečenog dela $m_1 = \rho_S \pi R_1^2, m_2 = \rho_S \pi R_2^2$.
- Superpozicija

$$I = I_1 - I_2 = \frac{1}{2} m_1 R_1^2 - \frac{1}{2} m_2 R_2^2$$

$$= \frac{m}{2} \frac{R_1^4 - R_2^4}{R_1^2 - R_2^2}$$

$$I_z = \frac{m}{2}(R_1^2 + R_2^2)$$

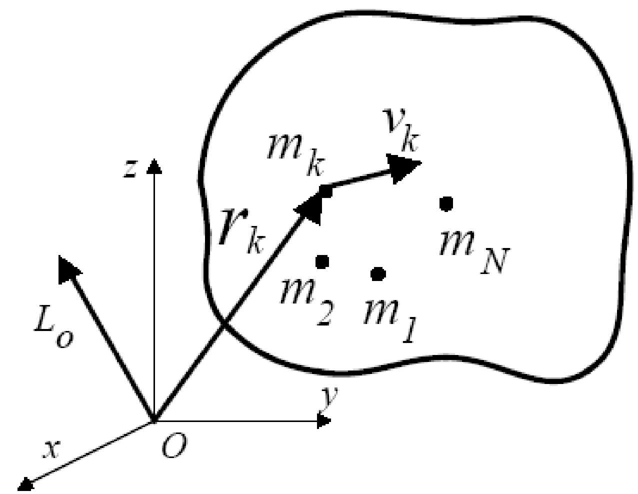
Teorema o promeni momenta količine kretanja sistema u odnosu na nepokretnu tačku

$$\vec{L}_O = \sum_k \vec{r}_k \times m_k \vec{v}_k \,.$$

$$\frac{d}{dt}(\vec{r}_k \times m_k \vec{v}_k) = \vec{r}_k \times \vec{F}^e_k + \vec{r}_k \times \vec{F}^i_k \,, \quad k = 1, ..., N.$$

$$\frac{d}{dt}\sum_{k=1}^{N} \vec{r}_k \times m_k \vec{v}_k = \sum_{k=1}^{N} \vec{r}_k \times \vec{F}^e_k + \sum_{k=1}^{N} \vec{r}_k \times \vec{F}^i_k \,. \qquad \sum_{k=1}^{N} \vec{r}_k \times \vec{F}^i_k = 0 \,,$$

$$\vec{M}^e_{O,R} \equiv \sum_{k=1}^{N} \vec{r}_k \times \vec{F}^e_k$$



$$\frac{d\vec{L}_O}{dt} = \sum_{k=1}^{N} \vec{r}_k \times \vec{F}^e_k \,, \qquad \frac{d\vec{L}_O}{dt} = \vec{M}^e_{O,R} \,.$$

Teorema o promeni momenta količine kretanja sistema

- Može se pokazati da prethodno izvedena teorema važi za osu koja prolazi kroz centar mase, kada se on kreće ubrzano, kao i za tačku nulte brzine

Prema centru mase:

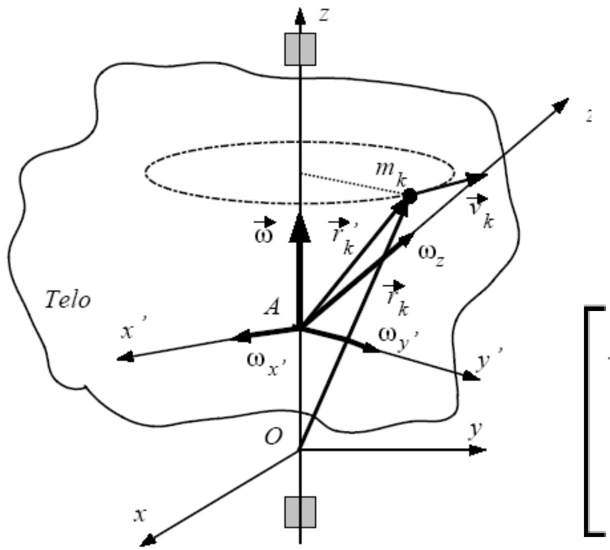
$$\frac{d\vec{L}_C}{dt} = \sum_k \vec{M}^e_{kC} \, .$$

$$\vec{M}^e_{kC} = \vec{r}_{kC} \times \vec{F}^e_k$$

Prema trenutnom polu nulte brzine:

$$\frac{d\vec{L}_P}{dt} = \vec{M}^e_{P,R},$$

Moment količine kretanja tela koje rotira oko fiksne ose (*dodatak*)



$$\vec{L}_A = [I]_A \vec{\omega} \,.$$

$$\begin{bmatrix} L_{x'} \\ L_{y'} \\ L_{z'} \end{bmatrix} = \begin{bmatrix} I_{x'x'} & I_{x'y'} & I_{x'z'} \\ I_{y'x'} & I_{y'y'} & I_{y'z'} \\ I_{z'x'} & I_{z'y'} & I_{z'z'} \end{bmatrix} \begin{bmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{bmatrix}$$

$$L_{x'} = \sum_k \Delta m_k (y_k'^2 + z_k'^2)\omega_{x'} - \sum_k \Delta m_k x_k' y_k' \omega_{y'} - \sum_k \Delta m_k x_k' z_k' \omega_{z'} \,,$$

Proizvodi inercije:

$$L_{y'} = -\sum_k \Delta m_k y_k' x_k' \omega_{x'} + \sum_k \Delta m_k (x_k'^2 + y_k'^2)\omega_{y'} - \sum_k \Delta m_k y_k' z_k' \omega_{z'} \,,$$

$$L_{z'} = -\sum_k \Delta m_k z_k' x_k' \omega_{x'} - \sum_k \Delta m_k z_k' y_k' \omega_{y'} + \sum_k \Delta m_k (x_k'^2 + y_k'^2)\omega_{z'} \,.$$

Zakon očuvanja momenta količine kretanja

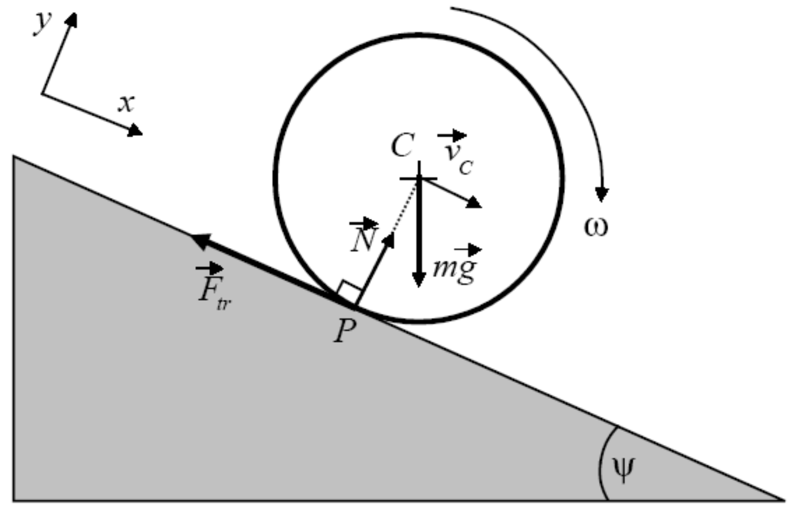
- Ako je rezultantni eksterni moment nula:
$$\vec{L} = const$$
- Primer: Klizačica (trenje zanemarljivo pa je rezultantni moment $M_z = 0$). Pri promeni figure:
$$I\vec{\omega} = const$$

$$I \uparrow \Longrightarrow \vec{\omega} \downarrow$$

$$I \downarrow \Longrightarrow \vec{\omega} \uparrow$$

Kotrljanje

- Šta se kotrlja brže?
- Za poređenje različitih oblika pogledati predavanja i vežbe.



Bez proklizavanja: $$a_C = \frac{g \sin \psi}{1 + \frac{I_C}{mR^2}}$$

- Minimalan koeficijent trenja tako da se disk kotrlja bez proklizavanja?

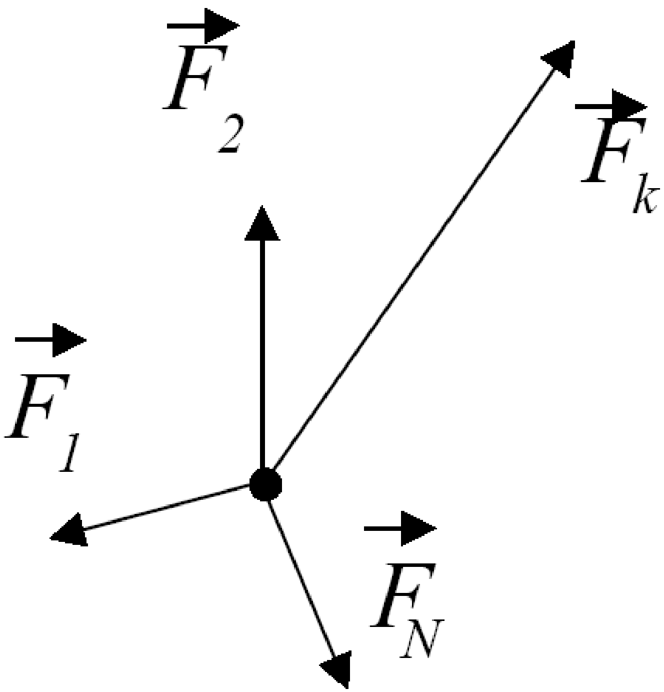
$$\mu_{\min} = \frac{1}{3} \mathrm{tg}(\psi).$$

- Kinetička energija:

$$E_k = E_{k,tran} + E_{k,rot} = \frac{1}{2} m v_C^2 + \frac{1}{2} I_C \omega^2 = \frac{1}{2} m (\omega R)^2 + \frac{1}{2} I_C \omega^2$$
$$= \frac{1}{2} (mR^2 + I_C) \omega^2 = \frac{1}{2} I_A \omega^2$$

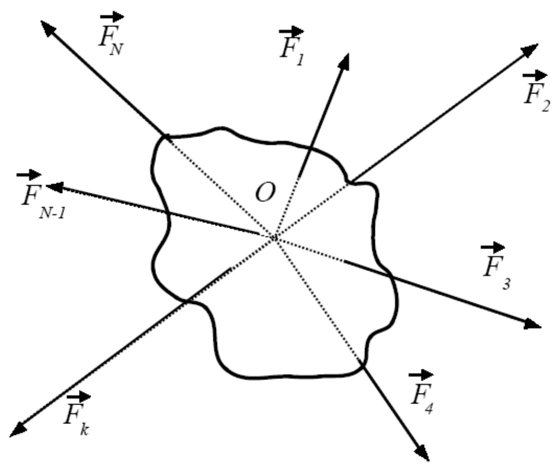
Ravnoteža (statika) materijalne tačke

- Relevantna je samo translacija



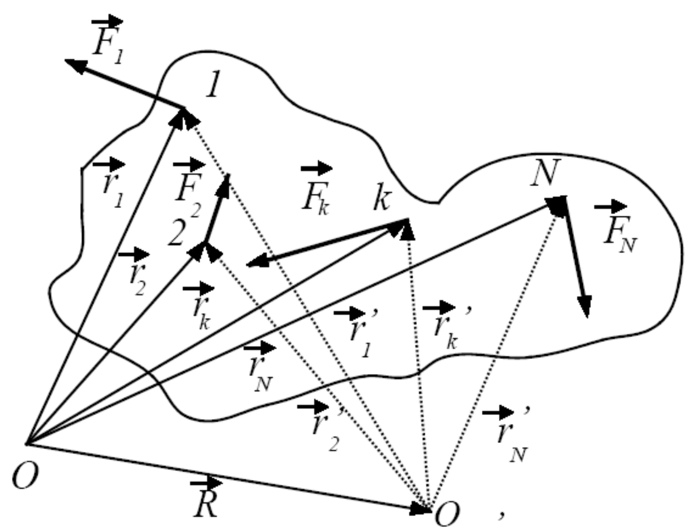
$$\vec{F}_{rez} = \sum_{k=1}^{N} \vec{F}_k = 0$$

Ravnoteža (statika) sistema MT i krutog tela



Transalcija

$$\vec{F}_{rez} = \sum_{k=1}^{N} \vec{F}_k = 0$$



Rotacija

$$\vec{M}_{rez} = \sum_{k=1}^{N} \vec{r}_k \times \vec{F}_k = 0$$

Hvala na pažnji!

- Kraj 8. časa!