Sistemi linearnih jednačina

Neka je dat sistem linearnih jednačina

$$\begin{gathered} a_{11}x_1 + ... + a_{1n}x_n = b_1 \\ \cdots \\ a_{m1}x_1 + ... + a_{mn}x_n = b_m \end{gathered}$$

pri čemu su svi koeficijenti iz *R*.

Matricu $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ nazivamo **matrica sistema**, a matricu $\overline{A} = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & \cdots & a_{mn} & b_n \end{array}\right]$

proširena matrica sistema.

Kroneker-Kapelijeva teorema: Sistem je saglasan akko je $rangA = rang\overline{A}$.

$$rang A \neq rang \bar{A} \quad \Leftrightarrow \quad \text{sistem je nemoguć}$$

$$rang A = rang \bar{A} \begin{cases} = n & \Leftrightarrow \quad \text{sistem je određen} \\ < n & \Leftrightarrow \quad \text{sistem je neodređen} \end{cases}$$

(*n* je broj promenljivih u sistemu)

1. Rešiti sistem linearnih jednačina u zavisnosti od vrednosti realnog parametra *p*:

1) $$\begin{aligned} x+3y+pz&=1 \\ -x+4y+6z&=0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 3 & p & 1 \\ -1 & 4 & 6 & 0 \end{array}\right] \hookleftarrow + = \left[\begin{array}{ccc|c} \boxed{1} & 3 & p & 1 \\ 0 & \boxed{7} & p+6 & 1 \end{array}\right] \quad \Rightarrow \quad rangA = rang\bar{A} = 2 < 3$$

Sistem je neodređen.

$$\begin{aligned} \boxed{x} + 3y + pz &= 1 \\ \boxed{7y} + (p+6)z &= 1 \end{aligned}$$ (nemarkiranu promenljivu zamenimo proizvoljnim parametrom)

$$z = \alpha$$

$$\Rightarrow \quad y = \frac{1-(p+6)\alpha}{7} \qquad \Rightarrow \qquad (x,y,z) = \left(\frac{18-4p}{7}\alpha + \frac{4}{7},\ \frac{1-(p+6)\alpha}{7},\ \alpha\right),$$

$$\alpha \in R$$

$$x = 1 - p\alpha - \frac{3(1-(p+6)\alpha)}{7} = \frac{18-4p}{7}\alpha + \frac{4}{7}$$

$$\boxed{\begin{aligned} 2)\qquad x+y-z+t&=2\\ 2x-y+z-t&=1\\ x+4y-4z+pt&=6 \end{aligned}}$$

$$\left[\begin{array}{cccc|c} \boxed{1} & 1 & -1 & 1 & 2\\ 2 & -1 & 1 & -1 & 1\\ 1 & 4 & -4 & p & 6 \end{array}\right] \begin{array}{l} |\cdot(-2)\ |\cdot(-1)\\ \hookleftarrow +\quad \swarrow + \end{array} \cong \left[\begin{array}{cccc|c} \boxed{1} & 1 & -1 & 1 & 2\\ 0 & \boxed{-3} & 3 & -3 & -3\\ 0 & 3 & -3 & p-1 & 4 \end{array}\right] \begin{array}{l} \\ \\ \hookleftarrow + \end{array} \cong \left[\begin{array}{cccc|c} \boxed{1} & 1 & -1 & 1 & 2\\ 0 & \boxed{-3} & 3 & -3 & -3\\ 0 & 0 & 0 & \boxed{p-4} & 1 \end{array}\right]$$

1^0 za $p=4$ $\quad rangA=2,\ rang\bar{A}=3$, pa sistem nema rešenja

2^0 za $p\neq 4$ $\quad rangA=rang\bar{A}=3<4$, pa je sistem neodređen

$$\begin{aligned} \boxed{x}+y-z+t&=2\\ \boxed{-3y}+3z-3t&=-3\\ \boxed{(p-4)t}&=1 \end{aligned} \Rightarrow\ z=\alpha,\ t=\frac{1}{p-4}\ ,$$

$$\boxed{-y}+\alpha-\frac{1}{p-4}=-1\ \Rightarrow\ y=\alpha+1-\frac{1}{p-4}$$

$$\boxed{x}+y-\alpha+\frac{1}{p-4}=2\ \Rightarrow\ x=1$$

$$\Rightarrow\ (x,y,z,t)=(1,\alpha+1-\frac{1}{p-4},\alpha,\frac{1}{p-4}),\ \alpha\in R$$

Ili:

$$x+y-z+t=2$$
$$2x-y+z-t=1$$
$$x+4y-4z+pt=6$$

$$\left[\begin{array}{cccc|c} 1 & \boxed{1} & -1 & 1 & 2 \\ 2 & -1 & 1 & -1 & 1 \\ 1 & 4 & -4 & p & 6 \end{array}\right] \begin{array}{c} |\cdot(-4) \\ \swarrow + \\ \end{array} \cong \left[\begin{array}{cccc|c} 1 & \boxed{1} & -1 & 1 & 2 \\ \boxed{3} & 0 & 0 & 0 & 3 \\ -3 & 0 & 0 & p-4 & -2 \end{array}\right] \cong \left[\begin{array}{cccc|c} 1 & \boxed{1} & -1 & 1 & 2 \\ \boxed{3} & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & \boxed{p-4} & 1 \end{array}\right]$$

1^0 za $p=4$ $\quad rangA=2,\ rang\overline{A}=3$, pa sistem nema rešenja

2^0 za $p\neq 4$ $\quad rangA=rang\overline{A}=3<4$, pa je sistem neodređen

$$x+\boxed{y}-z+t=2$$
$$\boxed{3x}=3 \Rightarrow x=1,\ t=\frac{1}{p-4},$$
$$\boxed{(p-4)t}=1$$

$$z=\alpha,\ 1+\boxed{y}-\alpha+\frac{1}{p-4}=2 \Rightarrow y=\alpha+1-\frac{1}{p-4},$$

$$\Rightarrow (x,y,z,t)=(1,\alpha+1-\frac{1}{p-4},\alpha,\frac{1}{p-4}),\ \alpha\in R$$

3)
$$\begin{aligned} x + y + z &= 2 \\ x+3y+(p+2)z &= -3p \\ x+(p+1)y+2z &= -2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 2 \\ 1 & 3 & p+2 & -3p \\ 1 & p+1 & 2 & -2 \end{array}\right] \begin{array}{l} |\cdot(-1) \\ \hookleftarrow + \\ \swarrow + \end{array} \cong \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 2 \\ 0 & \boxed{2} & p+1 & -3p-2 \\ 0 & p & 1 & -4 \end{array}\right] \begin{array}{l} \\ |\cdot(-p/2) \cong \\ \hookleftarrow + \end{array}$$

$$\cong \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 2 \\ 0 & \boxed{2} & p+1 & -3p-2 \\ 0 & 0 & \dfrac{-p^2-p+2}{2} & \dfrac{3p^2+2p-8}{2} \end{array}\right] = \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 2 \\ 0 & \boxed{2} & p+1 & -3p-2 \\ 0 & 0 & \boxed{\dfrac{-(p-1)(p+2)}{2}} & \dfrac{(3p-4)(p+2)}{2} \end{array}\right]$$

1^0 za $p=1$ $\quad rangA=2,\ rang\bar{A}=3$, pa sistem nema rešenja

2^0 za $p=-2$ $\quad rangA=2=rang\bar{A}<3$, pa je sistem neodređen

3^0 za $p\neq 1,\ p\neq -2$ $\quad rangA=rang\bar{A}=3$ sistem je određen

$$2^0 \quad \bar{A} \cong \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 2 \\ 0 & \boxed{2} & -1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{r} \boxed{x} + y + z = 2 \\ \boxed{2y} - z = 4 \end{array} \Rightarrow z = \alpha,\ y = 2 + \frac{\alpha}{2},\ x = 2 - 2 - \frac{\alpha}{2} - \alpha = -\frac{3\alpha}{2}$$

$$\Rightarrow \quad (x, y, z) = (-\frac{3\alpha}{2}, 2 + \frac{\alpha}{2}, \alpha), \quad \alpha \in R$$

$$3^0 \quad \bar{A} \cong \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 2 \\ 0 & \boxed{2} & p+1 & -3p-2 \\ 0 & 0 & \boxed{\frac{-(p-1)(p+2)}{2}} & \frac{(3p-4)(p+2)}{2} \end{array}\right] \Rightarrow \begin{array}{l} \boxed{x} + y + z = 2 \\ \boxed{2y} + (p+1)z = -3p-2 \\ \boxed{\frac{-(p-1)(p+2)}{2} z} = \frac{(3p-4)(p+2)}{2} \end{array}$$

$$\Rightarrow \quad z = \frac{3p-4}{1-p}, \qquad y = -3p - 2 - \frac{(p+1)(3p-4)}{1-p} = \frac{2}{1-p}, \quad x = 2 - \frac{2}{1-p} - \frac{3p-4}{1-p} = \frac{4-5p}{1-p}$$

$$\Rightarrow \quad (x, y, z) = (\frac{4-5p}{1-p}, \frac{2}{1-p}, \frac{3p-4}{1-p})$$

4)
$$\begin{aligned} x + y + z &= 3 \\ x - py + 2z &= 1 \\ -2x + 2y - pz &= -4 \end{aligned}$$

$$\bar{A} = \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 1 & -p & 2 & 1 \\ -2 & 2 & -p & -4 \end{array}\right] \begin{array}{l} |\cdot(-1) \quad |\cdot 2 \\ \hookleftarrow + \quad \swarrow + \\ \\ \end{array} \cong \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 0 & -p-1 & \boxed{1} & -2 \\ 0 & 4 & -p+2 & 2 \end{array}\right] \begin{array}{l} \\ |\cdot(p-2) \\ \hookleftarrow + \end{array} \cong$$

$$\cong \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 0 & -p-1 & \boxed{1} & -2 \\ 0 & -p^2+p+6 & 0 & 6-2p \end{array}\right] = \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 0 & -p-1 & \boxed{1} & -2 \\ 0 & \boxed{-(p-3)(p+2)} & 0 & -2(p-3) \end{array}\right]$$

1^0 za $p = -2$ $\quad rangA = 2,\ rang\bar{A} = 3$, pa sistem nema rešenja

2^0 za $p = 3$ $\quad rangA = 2 = rang\bar{A} < 3$, pa je sistem neodređen

3^0 za $p \neq 3,\ p \neq -2$ $\quad rangA = rang\bar{A} = 3$, pa je sistem određen

2^0

$$\begin{aligned} &\boxed{x} + y + z = 3 \\ &(-4)y + \boxed{z} = -2 \end{aligned} \Rightarrow y = \alpha,\ z = -2 + 4\alpha,\ x = 3 - \alpha + 2 - 4\alpha = 5 - 5\alpha \Rightarrow$$

$$\Rightarrow (x, y, z) = (5 - 5\alpha, \alpha, -2 + 4\alpha) \ ,\ \alpha \in R$$

3^0

$$\begin{aligned} &\boxed{x} + y + z = 3 \\ &-(p+1)y + \boxed{z} = -2 \\ &\boxed{-(p-3)(p+2)y} = -2(p-3) \end{aligned}$$

$$\Rightarrow y = \frac{2}{p+2},\ z = -2 + \frac{2(p+1)}{p+2} = -\frac{2}{p+2},\ x = 3 - \frac{2}{p+2} + \frac{2}{p+2} = 3 \Rightarrow$$

$$\Rightarrow (x, y, z) = (3, \frac{2}{p+2}, -\frac{2}{p+2})$$

$$5)\quad \begin{aligned} x + y + z &= 3 \\ 2x + py + z &= 6 \\ px + 2y - z &= 3 \end{aligned}$$

$$\bar{A} = \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 2 & p & 1 & 6 \\ p & 2 & -1 & 3 \end{array}\right] \begin{array}{l} |\cdot(-2) \quad |\cdot(-p) \\ \hookleftarrow + \quad \swarrow + \\ \end{array} \cong \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 0 & \boxed{p-2} & -1 & 0 \\ 0 & 2-p & -1-p & 3-3p \end{array}\right] \hookleftarrow +$$

$$1^0 \quad \text{za } p = 2, \quad \bar{A} \cong \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 0 & 0 & \boxed{-1} & 0 \\ 0 & 0 & -3 & -3 \end{array}\right] \begin{array}{l} \\ |\cdot(-3) \\ \hookleftarrow + \end{array} \cong \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 0 & 0 & \boxed{-1} & 0 \\ 0 & 0 & 0 & \boxed{-3} \end{array}\right], \text{ tj. } rangA = 2, \ rang\bar{A} = 3 \text{ i}$$

$rangA \neq rang\bar{A}$, pa sistem nema rešenja

2^0 za $p \neq 2$, $\bar{A} \cong \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 0 & \boxed{p-2} & -1 & 0 \\ 0 & 2-p & -1-p & 3-3p \end{array}\right] \begin{matrix} \\ \\ \hookleftarrow + \end{matrix} \cong \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 0 & \boxed{p-2} & -1 & 0 \\ 0 & 0 & \boxed{-2-p} & 3-3p \end{array}\right]$, pa važi:

A. za $p=-2$, $\bar{A} \cong \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 0 & \boxed{-4} & -1 & 0 \\ 0 & 0 & 0 & \boxed{-3} \end{array}\right]$, takođe je $rangA = 2$, $rang\bar{A} = 3$ i $rangA \neq rang\bar{A}$ i

sistem nema rešenja

B. za $p \neq -2$,

$$\begin{aligned} \boxed{x} + y + z &= 3 \\ \boxed{(p-2)y} - z &= 0 \\ \boxed{(-2-p)z} &= 3-3p \end{aligned} \quad \Rightarrow \quad z = \frac{3(p-1)}{p+2},\ y = \frac{3(p-1)}{p^2-4},\ x = 3 - \frac{3(p-1)}{(p-2)(p+2)} - \frac{3(p-1)}{p+2} = \frac{2p-5}{p^2-4}$$

tj. $(x,y,z) = (\frac{2p-5}{p^2-4}, \frac{3(p-1)}{p^2-4}, \frac{3(p-1)}{p+2})$.

$$\boxed{\begin{aligned} 6)\quad x + y + pz &= 1 - p \\ px - y + z &= -1 \\ x - py - z &= 0 \end{aligned}}$$

$$\bar{A} = \left[\begin{array}{ccc|c} \boxed{1} & 1 & p & 1-p \\ p & -1 & 1 & -1 \\ 1 & -p & -1 & 0 \end{array}\right] \begin{array}{l} |\cdot(-p) \quad |\cdot(-1) \\ \hookleftarrow + \quad \swarrow + \\ \; \end{array} \cong \left[\begin{array}{ccc|c} \boxed{1} & 1 & p & 1-p \\ 0 & \boxed{-1-p} & 1-p^2 & -1-p+p^2 \\ 0 & -p-1 & -1-p & -1+p \end{array}\right]$$

1^0 za $p=-1$, biće $\bar{A} \cong \left[\begin{array}{ccc|c} \boxed{1} & 1 & p & 1-p \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{array}\right]$, tj. $rangA = 1$, $rang\bar{A} = 2$ i $rangA \neq rang\bar{A}$ i

sistem nema rešenja

2^0 za $p \neq -1$

$$\bar{A} \cong \left[\begin{array}{ccc|c} \boxed{1} & 1 & p & 1-p \\ 0 & \boxed{-1-p} & 1-p^2 & -1-p+p^2 \\ 0 & -p-1 & -1-p & -1+p \end{array}\right] \begin{array}{l} \\ |\cdot(-1) \\ \hookleftarrow + \end{array} \cong \left[\begin{array}{ccc|c} \boxed{1} & 1 & p & 1-p \\ 0 & \boxed{-1-p} & 1-p^2 & -1-p+p^2 \\ 0 & 0 & -2-p+p^2 & 2p-p^2 \end{array}\right] =$$

$$= \left[\begin{array}{ccc|c} \boxed{1} & 1 & p & 1-p \\ 0 & \boxed{-1-p} & 1-p^2 & -1-p+p^2 \\ 0 & 0 & \boxed{(p-2)(p+1)} & -p(p-2) \end{array}\right]$$ imamo dva podslučaja:

A. za $p=2$, biće $\bar{A} \cong \left[\begin{array}{ccc|c} \boxed{1} & 1 & 2 & -1 \\ 0 & \boxed{-3} & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$, tj. $\begin{array}{l} \boxed{x}+y+2z=-1 \\ \boxed{-3y}-3z=1 \end{array} \Rightarrow z=\alpha,\ y=-\frac{1}{3}-\alpha$,

$x=-1+\frac{1}{3}+\alpha-2\alpha=-\alpha-\frac{2}{3}$, pa je $(x,y,z)=(-\alpha-\frac{2}{3},-\frac{1}{3}-\alpha,\alpha)$, $\alpha \in R$

B. za $p \neq 2$, rešavaćemo sistem

$$\boxed{x} + y + pz = 1 - p$$
$$\boxed{(-1-p)y} + (1-p^2)z = -1 - p + p^2$$
$$\boxed{(p-2)(p+1)z} = -p(p-2)$$

, pa dobijamo $(x, y, z) = (-\frac{2p}{p+1}, \frac{1+2p}{1+p}, -\frac{p}{p+1})$

7)
$$\begin{aligned} x + y + z &= 1 \\ x - 2y + 2z &= p \\ px - y + 3z &= 3 \\ 3x + 2pz &= 4 \end{aligned}$$

$$\bar{A} = \left[\begin{array}{ccc|c} 1 & \boxed{1} & 1 & 1 \\ 1 & -2 & 2 & p \\ p & -1 & 3 & 3 \\ 3 & 0 & 2p & 4 \end{array}\right] \begin{array}{l} |\cdot 2 \\ \hookleftarrow + \quad \swarrow + \\ \\ \\ \end{array} \cong \left[\begin{array}{ccc|c} 1 & \boxed{1} & 1 & 1 \\ 3 & 0 & \boxed{4} & p+2 \\ p+1 & 0 & 4 & 4 \\ 3 & 0 & 2p & 4 \end{array}\right] \begin{array}{l} \\ |\cdot(-1) \ |\cdot(-p/2) \\ \hookleftarrow + \quad \swarrow + \\ \\ \end{array} \cong$$

$$\cong \left[\begin{array}{ccc|c} 1 & \boxed{1} & 1 & 1 \\ 3 & 0 & \boxed{4} & p+2 \\ \boxed{p-2} & 0 & 0 & 2-p \\ \frac{6-3p}{2} & 0 & 0 & \frac{8-p^2-2p}{2} \end{array}\right]$$

1^0 za $p=2$, $\bar{A}\cong\left[\begin{array}{ccc|c}1 & \boxed{1} & 1 & 1\\ 3 & 0 & \boxed{4} & 4\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$, pa rešavamo sistem $\begin{array}{l}x+\boxed{y}+z=1\\ 3x+\boxed{4z}=4\end{array}$ i dobijamo $x=\alpha$,

$$z=1-\frac{3\alpha}{4} \text{ i } y=1-\alpha-1+\frac{3\alpha}{4}=-\frac{\alpha}{4}\text{, tj. } (x,y,z)=(\alpha,-\frac{\alpha}{4},1-\frac{3\alpha}{4}),\ \alpha\in R$$

2^0 za $p\neq 2$,

$$\bar{A}\cong\left[\begin{array}{ccc|c}1 & \boxed{1} & 1 & 1\\ 3 & 0 & \boxed{4} & p+2\\ \boxed{p-2} & 0 & 0 & 2-p\\ \frac{6-3p}{2} & 0 & 0 & \frac{8-p^2-2p}{2}\end{array}\right]\begin{array}{l} \\ \\ |\cdot 3/2\\ \hookleftarrow +\end{array}\cong\left[\begin{array}{ccc|c}1 & \boxed{1} & 1 & 1\\ 3 & 0 & \boxed{4} & p+2\\ \boxed{p-2} & 0 & 0 & 2-p\\ 0 & 0 & 0 & \frac{14-p^2-5p}{2}\end{array}\right]$$

$$\overline{A} \cong \left[\begin{array}{ccc|c} 1 & \boxed{1} & 1 & 1 \\ 3 & 0 & \boxed{4} & p+2 \\ \boxed{p-2} & 0 & 0 & 2-p \\ 0 & 0 & 0 & \boxed{\dfrac{-(p+7)(p-2)}{2}} \end{array}\right]$$

A. za $p \neq -7$, $rangA = 3$, $rang\overline{A} = 4$, pa sistem nema rešenja

B. za $p = -7$, $rangA = 3 = rang\overline{A}$, pa je sistem određen i rešenja nalazimo rešavajući:

$$\begin{aligned} x + \boxed{y} + z &= 1 \\ 3x + \boxed{4z} &= -5 \\ \boxed{-9x} &= 9 \end{aligned}$$

, odakle dobijamo $(x, y, z) = (-1, \frac{5}{2}, -\frac{1}{2})$.